REGULARITY PROPERTIES FOR FAMILIES OF MEASURES ON A METRIC SPACE

M. ANN PIECH

Let $\{\mu(x, dy): x \in S\}$ be a family of finite real-valued signed Borel measures on a metric space S —i.e. $|\mu(x, E)| < \infty$ for each $x \in S$ and for each Borel set E in S . Denote the metric on S by d . Let $\mathfrak{B}(S)$ denote the family of real-valued bounded measurable functions on S , $\mathfrak{C}(S)$ the continuous functions in $\mathfrak{B}(S)$, and $\mathfrak{A}(S)$ the functions in $\mathfrak{C}(S)$ which satisfy a Lip-1 condition—($f \in \mathfrak{A}(S) \Leftrightarrow f \in \mathfrak{B}(S)$ and $\exists$ a constant c such that $|f(x) - f(y)| \leq c \cdot d(x, y)$ for all $x, y \in S$). Let $\|f\|_\infty = \sup_{x \in S} |f(x)|$ for each $f \in \mathfrak{B}(S)$.

We may now define

$$Mf(x) = \int_S f(y) \mu(x, dy)$$

for each Borel function f on S for which the integral exists. By the Hahn Decomposition Theorem, $\int_S |\mu|(x, dy) < \infty$ for each $x \in S$. Thus $Mf(x)$ exists for each $x \in S$ whenever $f \in \mathfrak{B}(S)$. Moreover $|Mf(x)| \leq \|f\|_\infty \int_S |\mu|(x, dy)$. We would like to know if Mf is measurable and, if so, if M defines a bounded linear operator on $\mathfrak{B}(S)$ (considering $\mathfrak{B}(S)$ as a Banach space with the sup norm).

In potential theory on Hilbert space [1], [2], families of measures of the above form often arise which possess the property that $f \in \mathfrak{A}(S) \Rightarrow Mf \in \mathfrak{C}(S)$. In some important cases, the measures are all nonnegative. We will investigate the regularity imparted to M and to the family $\{\mu(x, dy)\}$ by assumptions of this nature. Since the context in which this problem arises is one where the underlying space is not locally compact, the measures are not translation invariant, and the norm (or metric) is not differentiable, we wish to avoid any techniques which would require these additional assumptions. In fact, the only assumption on S is that it be a metric space.

PROPOSITION 1. *If $M: \mathfrak{A}(S) \rightarrow \mathfrak{B}(S)$, then if E is either an open or a closed set in S , $\mu(x, E)$ is measurable as a function of x .*

PROPOSITION 2. *If $M: \mathfrak{A}(S) \rightarrow \mathfrak{C}(S)$, and $\mu(x, E) \geq 0$ (≤ 0) for all $x \in S$ and all Borel sets $E \subset S$, then $\mu(x, F)$ is upper (lower) semicontinuous in x for each closed set $F \subset S$. For each open set $G \subset S$, $\mu(x, G)$ is lower (upper) semicontinuous in x .*

Received by the editors June 25, 1969.

PROOF OF PROPOSITION 1. Let F be a closed set in S . Define

$$f_n(x) = \exp[-n\delta(x, F)]$$

where $\delta(x, F)$ is the distance from x to F . Since for any two nonnegative real numbers a and b

$$e^{-na} - e^{-nb} = -ne^{-nc}(a - b)$$

where c is some number between a and b , we have

$$|e^{-na} - e^{-nb}| \leq n|a - b|.$$

Thus for $x, y \in S$,

$$\begin{aligned} |f_n(x) - f_n(y)| &\leq n|\delta(x, F) - \delta(y, F)| \\ &\leq nd(x, y) \end{aligned}$$

and so $f_n \in \mathcal{A}(S)$ for each $n = 1, 2, \dots$. $f_n(x) \searrow \chi_F(x)$; hence $Mf_n(x) \rightarrow M\chi_F(x) = \mu(x, F)$ for each $x \in S$. The measurability of each $Mf_n(x)$ implies the measurability of $\mu(x, F)$.

If G is open in S , then $\mu(x, G) = \mu(x, S) - \mu(x, \mathcal{C}G)$. S and $\mathcal{C}G$ are closed, so $\mu(x, S)$ and $\mu(x, \mathcal{C}G)$ are measurable, and we conclude that $\mu(x, G)$ is measurable.

PROOF OF PROPOSITION 2. Assume that $\mu(x, dy)$ is nonnegative for each $x \in S$. If F is closed in S , then, defining $\{f_n\}$ as in the previous proof, we have $Mf_n(x) \searrow M\chi_F(x) = \mu(x, F)$. The continuity of each Mf_n now implies the upper semicontinuity of $\mu(x, F)$. If G is open in S , then $\mu(x, G) = \mu(x, S) - \mu(x, \mathcal{C}G)$. Since $\mu(x, \mathcal{C}G)$ is upper semicontinuous, it will follow that $\mu(x, G)$ is lower semicontinuous if (and, trivially, only if) we show that $\mu(x, S)$ is continuous. But if 1 denotes the function which is identically 1 on S , then $1 \in \mathcal{A}(S)$, and $\mu(x, S) = M1(x) \in \mathcal{C}(S)$.

When the family $\{\mu(x, dy)\}$ is nonpositive, then $Mf_n(x) \nearrow \mu(x, F)$ and the lower semicontinuity of $\mu(x, F)$, and hence the upper semicontinuity of $\mu(x, G)$, follow.

COROLLARY 1.1. If $\{\mu(x, dy): x \in S\}$ and $\{\mu'(x, dy): x \in S\}$ are two families of finite Borel measures such that

$$\int_S f(y) \mu(x, dy) = \int_S f(y) \mu'(x, dy)$$

for all $f \in \mathcal{A}(S)$, then $\mu(x, E) = \mu'(x, E)$ for all Borel sets $E \subset S$ and for all $x \in S$.

PROOF. Set $\mathcal{S} = \{ \text{Borel sets } E \subset S : \mu(x, E) = \mu'(x, E) \text{ for all } x \in S \}$. If F is closed in S and if $\{f_n\}$ is defined as in Proposition 1, then

$$\begin{aligned} 0 &= \int_S f_n(y) \mu(x, dy) - \int_S f_n(y) \mu'(x, dy) \\ &\rightarrow \mu(x, F) - \mu'(x, F), \quad \text{and so } F \in \mathcal{S}. \end{aligned}$$

If E belongs to the field generated by the closed sets, then E may be represented as a disjoint union, $\bigcup_{i=1}^m (F_i \cap G_i)$ where each F_i is closed in S and each G_i is open in S . From the proof of Proposition 1, there exist sequences $\{f_n^i\}$ and $\{g_n^i\}$ ($n=1, 2, \dots$; $i=1, 2, \dots, m$) of members of $\mathcal{Q}(S)$ such that $f_n^i \rightarrow \chi_{F_i}$ and $g_n^i \rightarrow \chi_{G_i}$ pointwise and boundedly for each $i=1, 2, \dots, m$. Since $\mathcal{Q}(S)$ is an algebra, it follows that there exists a sequence $\{h_n\}$ of members of $\mathcal{Q}(S)$ such that $h_n \rightarrow \chi_E$ pointwise and boundedly. Thus

$$\begin{aligned} \mu(x, E) &= \lim_{n \rightarrow \infty} \int_S h_n(y) \mu(x, dy) \\ &= \lim_{n \rightarrow \infty} \int_S h_n(y) \mu'(x, dy) \\ &= \mu'(x, E), \end{aligned}$$

and so $E \in \mathcal{S}$. If $E_n \nearrow (\searrow) E$ and $E_n \in \mathcal{S}$ ($n=1, 2, \dots$), then $\mu(x, E) = \lim_{n \rightarrow \infty} \mu(x, E_n) = \lim_{n \rightarrow \infty} \mu'(x, E_n) = \mu'(x, E)$. Thus $\mathcal{S}$ contains the field generated by the closed sets and is closed under monotone limits. It follows that $\mathcal{S}$ is the Borel σ -field of S .

For the next proposition, we will assume that the measures are all nonnegative. We call E an absolute null set relative to $\{\mu(x, dy) : x \in S\}$ if $\mu(x, E) = 0$ for all $x \in S$.

PROPOSITION 3. Assume that $M: \mathcal{Q}(S) \rightarrow \mathcal{C}(S)$. If $A \subset S$ and if ∂A is an absolute null set relative to $\{\mu(x, dy)\}$, then $\mu(x, A)$ is continuous.

PROOF. $A/A^0 \subset \partial A$ and $\bar{A}/A \subset \partial A$. Therefore

$$\begin{aligned} \mu(x, A) &= \mu(x, A^0) + \mu(x, A/A^0) \\ &= \mu(x, A^0), \quad \text{which is lower semicontinuous.} \end{aligned}$$

Also

$$\begin{aligned} \mu(x, A) &= \mu(x, \bar{A}) - \mu(x, \bar{A}/A) \\ &= \mu(x, A), \quad \text{which is upper semicontinuous.} \end{aligned}$$

It follows that $\mu(x, A)$ is continuous.

REMARK 3.1. We cannot expect $\mu(x, A)$ to be continuous in general. For if $p_t(dy)$ is Wiener measure with variance parameter t (for any $t > 0$) on Wiener space B , and if we define $p_t(x, E) = p_t(E - x)$ for any Borel set E in B , then in Gross [1] it is shown that given any $x_0 \in B$ there exists a closed set A such that $p_t(x, A)$ is discontinuous at $x = x_0$.

PROPOSITION 4. *If $M: \mathfrak{A}(S) \rightarrow \mathfrak{B}(S)$, then for each $f \in \mathfrak{B}(S)$, Mf is measurable.*

PROOF. Set $\mathfrak{s} = \{E: M\chi_E(x) \text{ exists and is measurable}\}$. $\mathfrak{s}$ contains the open and the closed sets in S (by Proposition 1). If E belongs to the field generated by the closed sets, and if we construct $\{h_n\}$ as in the proof of Corollary 1.1, then Mh_n is measurable for each n and $Mh_n \rightarrow M\chi_E$. Thus $E \in \mathfrak{s}$. Since $\mathfrak{s}$ is obviously closed under monotone limits, it follows that $\mathfrak{s}$ is the Borel σ -field of S . Now the set of all f such that $Mf(x)$ exists and is measurable contains the characteristic functions of the Borel sets and is closed under bounded pointwise limits. Thus it includes all $f \in \mathfrak{B}(S)$.

PROPOSITION 5. *If $M: \mathfrak{A}(S) \rightarrow \mathfrak{B}(S)$, then M defines a bounded linear operator on $\mathfrak{B}(S)$ if and only if there exists a constant C such that*

$$(*) \quad \int_S |\mu|(x, dy) \leq C \quad \text{for all } x \in S.$$

PROOF. Assume first that $(*)$ holds. Then $|Mf(x)| \leq C\|f\|_\infty$ for all $x \in S$, so $\|Mf\|_\infty \leq C\|f\|_\infty$ and $\|M\| \leq C$.

Conversely, assume M defines a bounded linear operator on $\mathfrak{B}(S)$. For each $x \in S$, define E_x^+ and E_x^- so that $E_x^+ \cap E_x^- = \emptyset$, $E_x^+ \cup E_x^- = S$, and

$$\begin{aligned} \int_{E_x^+} \mu(x, dy) &= \int_{E_x^+} |\mu|(x, dy) \\ \int_{E_x^-} \mu(x, dy) &= - \int_{E_x^-} |\mu|(x, dy). \end{aligned}$$

(These sets exist, by the Hahn Decomposition Theorem.) Define

$$\begin{aligned} f^x(y) &= 1 \quad \text{if } y \in E_x^+, \\ &= -1 \quad \text{if } y \in E_x^-. \end{aligned}$$

Then $f^x \in \mathfrak{B}(S)$, $\|f^x\|_\infty = 1$, and $\int_S f^x(y) \mu(x, dy) = \int_S |\mu|(x, dy)$. Since M is bounded, we must have $\int_S |\mu|(x, dy) \leq \|M\|$ for each $x \in S$.

REMARK 5.1. M is a bounded linear operator on $\mathfrak{B}(S)$ if and only if M is a linear operator on $\mathfrak{B}(S)$. For if we assume that M is linear, and for each $x \in S$ define $M_x: \mathfrak{B}(S) \rightarrow R$ by $M_x f = Mf(x)$, then each M_x is a bounded linear transformation ($\|M_x\| \leq \int_S |\mu|(x, dy)$). Moreover $\|M_x f\| \leq \|Mf\|_\infty$ for all $x \in S$. Thus, by the Principle of Uniform Boundedness, $\{\|M_x\|: x \in S\}$ is bounded—i.e. there exists a constant K such that $\|Mf(x)\| = \|M_x f\| \leq \|M_x\| \|f\|_\infty \leq K \|f\|_\infty$ for all $f \in \mathfrak{B}(S)$.

EXAMPLE 5.2. Let S be the real line with the standard metric, and h be a fixed, nonzero real number. Set

$$\begin{aligned}\mu(x, dy) &= |x| [\delta_{x+h/|x|}(dy) - \delta_x(dy)] & x \neq 0, \\ \mu(0, E) &= 0 & \text{for all Borel sets } E \text{ in } S.\end{aligned}$$

Here δ_a is Dirac measure concentrated at a . For $f \in \mathfrak{B}(S)$ we have

$$\begin{aligned}Mf(x) &= |x| [f(x + h/|x|) - f(x)] & x \neq 0, \\ &= 0 & x = 0.\end{aligned}$$

If we put the Lip-1 norm on $\mathfrak{A}(S)$ ($\|f\|_1 = \|f\|_\infty + \inf\{c: |f(x) - f(y)| \leq c|x - y| \text{ for all } x, y \in S\}$), then $f \rightarrow Mf$ is a bounded linear map from $\mathfrak{A}(S)$ to $\mathfrak{B}(S)$. Since $\int_S |\mu|(x, dy) = 2|x|$, $(*)$ fails to hold.

REFERENCES

1. L. Gross, *Potential theory on Hilbert space*, J. Functional Analysis **1** (1967), 123–181. MR **37** #3331.
2. M. Ann Piech, *A fundamental solution of the parabolic equation on Hilbert space*, J. Functional Analysis **3** (1969), 85–115.

STATE UNIVERSITY OF NEW YORK AT BUFFALO