COMMUTATIVE QF-1 ARTINIAN RINGS ARE QF

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ABSTRACT. In a recent paper, D. R. Floyd proved several results on algebras, each of whose faithful representations is its own bicommutant (=R. M. Thrall's QF-1 algebras, a generalization of QF-algebras) among which was the theorem in the title for algebras. We obtain our extension of Floyd's result by use of interlacing modules, replacing his arguments involving the representations themselves.

In [10], Thrall observed that the class of finite-dimensional algebras over which every faithful representation has the double centralizer property (i.e., is its own bicommutant) properly contains the class of quasi-Frobenius (=QF) algebras. He called the members of the former class QF-1 algebras and posed the intriguing problem of characterizing these algebras in terms of ideal structure. Solutions for this problem have been given for generalized uniserial algebras [5] and for commutative algebras [4]. Every faithful module M over a QF ring R has the double centralizer property in the sense that the natural homomorphism $\lambda: R \to \mathrm{Hom}_C(M, M)$ (where $C = \mathrm{Hom}_R(M, M)$) is onto (see [2, §59]). Thus Thrall's definition and his problem extend naturally to QF-1 artinian rings.

In view of recent results on the dominant dimension of an artinian ring (see [1, Theorem 2] or [8, Lemma 9]), the characterization of QF-1 generalized uniserial algebras (and its proof) given in [5] remains valid for generalized uniserial rings. In this note we prove the theorem of the title, thus extending a theorem that Floyd proved for finite-dimensional algebras by means of matrix representations [4].

It is not difficult to show that a direct sum of rings is QF or QF-1 if and only if so is each of the direct summands. Thus for our purposes we may assume that R is a commutative local artinian ring. According to Nakayama's original definition [9, p. 8], such a ring is QF if and only if its *R-socle* (i.e., its largest semisimple R-submodule) $S(R) = S({}_RR)$ is simple. We shall prove the theorem by constructing, in the event that $S(R)$ is not simple, a faithful module whose double centralizer has an R-socle larger than that of R. The methods used in this

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construction are suggested by the Lemma of [7] and Theorem 3.1 of [3].

Let R be a commutative local artinian ring and let M be a finitely generated indecomposable R-module with centralizer $C = \mathrm{Hom}_R(M, M)$ and double centralizer $C' = \mathrm{Hom}_C(M, M)$. Let $K = R/\mathrm{Rad}\, R$ and $D = C/\mathrm{Rad}\, C$. In this setting we have the following.

(1) The ring C is completely primary, in the sense that D is a division ring and Rad C is nilpotent [6, Chapter 4].

(2) Since R is commutative, C and C' are algebras over R, via

$$(r\gamma)(m) = r\gamma(m) = \gamma(rm) = (\gamma r)(m)$$

for $r \in R$, $\gamma \in \mathrm{Hom}(M, M)$, $m \in M$.

(3) (Rad R)C is a nilpotent ideal in C, so D is an algebra over the field K.

(4) The C-socle of M, $S({}_CM)$, is an R-C-module annihilated by Rad C and hence is a K-D-vector space.

With these observations we can now prove that the R-socle of C', $S({}_RC')$, has length at least as large as the dimension of $S({}_CM)$ over K. That is,

$$|S({}_RC'):K| \geqq |S({}_CM):K|. \tag{5}$$

PROOF. Let T be a maximal C-submodule of M. Then, because the functor $\mathrm{Hom}_C(\ ,\ )$ is left exact in both variables and C is an R-algebra, there is an R-monomorphism

$$0 \to \mathrm{Hom}_C(M/T, S({}_CM)) \to C'.$$

But since Rad C (and hence Rad R) annihilates M/T and $S({}_CM)$ this is really a K-monomorphism

$$0 \to \mathrm{Hom}_D(M/T, S({}_CM)) \to S({}_RC').$$

Thus, since M/T is a one-dimensional D-space and $S({}_CM)$ is a finite-dimensional D-space, we have

$$\begin{aligned} |S({}_RC'):K| &\geqq |\mathrm{Hom}_D(M/T, S({}_CM)):K| \\ &= |S({}_CM):D|\,|D:K| = |S({}_CM):K|. \end{aligned}$$

Now we are in a position to prove our main result.

THEOREM. *Every commutative artinian* QF-1 *ring is* QF.

PROOF. Suppose that R is as above and has distinct (but necessarily isomorphic) minimal ideals S and S'. Let $\phi: S \to S'$ be an isomorphism and form the interlacing module

$$M = (R \times R)/L, \qquad L = \{(s, -\phi(s)) \mid s \in S\}.$$

Then M contains a copy of R and so is faithful. Suppose γ is an R-endomorphism of M. If $\eta: R\times R \to M$ is the natural epimorphism then, using the projectivity of $R\times R$, one obtains an R-map $\bar{\gamma}$ making the diagram

$$\begin{array}{ccc} R \times R & \xrightarrow{\bar{\gamma}} & R \times R \\ \eta \downarrow & & \downarrow \eta \\ M & \xrightarrow{\gamma} & M \end{array}$$

commute and consequently taking L into L. The operation of $\bar{\gamma}$ on $R\times R$ is just that of some matrix

$$\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}, \qquad r_{ij} \in R.$$

The stability of L under $\bar{\gamma}$ yields for each $s \in S$, an $\bar{s} \in S$ such that s and $\bar{s}$ satisfy the matrix equation

$$(s, -\phi(s)) \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} = (\bar{s}, -\phi(\bar{s})).$$

From these equations and the independence of S and S' it follows easily that r_{12} and r_{21} both annihilate S and so are nilpotent, and that r_{11} and r_{22} are simultaneously either nilpotent or invertible. Thus the matrix of $\bar{\gamma}$ either has all nilpotent entries or is invertible. But γ is nilpotent or invertible if $\bar{\gamma}$ is, by the commutativity of the diagram, so M is indecomposable. Moreover, if γ is nilpotent then $\bar{\gamma}$, having a matrix with radical entries, must annihilate $S(R)\times S(R)$. That is,

$$\gamma(\eta(S(R) \times S(R))) = \eta(\bar{\gamma}(S(R) \times S(R))) = 0$$

whenever, in our earlier notation, $\gamma \in \operatorname{Rad} C$. This proves that

$$\eta(S(R) \times S(R)) \subseteq S({}_C M)$$

and the containment is as K-spaces. Now surely

$$\left| \eta(S(R) \times S(R)) : K \right| = 2 \left| S(R) : K \right| - 1,$$

so by (5) the double centralizer C' of M must have an R-socle strictly larger than that of R. This completes the proof.

If R is a semisimple ring then all R-modules (faithful or not) have the double centralizer property. Thus one wonders which rings satisfy this condition. Recalling that a ring is uniserial if and only if each of its factor rings is QF (see [5]), we obtain the

COROLLARY. *Every module over a commutative artinian ring R has the double centralizer property if and only if R is a uniserial ring.*

ADDED IN PROOF. V. P. Camillo has independently obtained these results using different methods.

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