GENERATORS OF S_α

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In this paper we will discuss the algebras of functions $f(z)=\sum_{n=0}^{\infty} a_n z^n$ such that $\sum_{n=0}^{\infty}(n+1)^\alpha |a_n| < \infty$ with $\|f\|_\alpha = \sum_{n=0}^{\infty}(n+1)^\alpha |a_n|$, $\alpha \geqq 1$. When $\alpha=0$ we have the algebra l_1 (which we will henceforth denote by l) about which it is conjectured that the obvious necessary condition for f to generate, i.e. that f separate points, is also sufficient, but thus far it has been proven only that it is sufficient if f separates points and $f' \in H^1$ [**1**].

We show here that if $\alpha \geqq 1$ then the obvious necessary conditions for f to generate, i.e. that f separate points and $f' \neq 0$ in the closed unit disc, are also sufficient.

DEFINITION. S_α is the Banach space of analytic functions $f(z)=\sum_{n=0}^{\infty} a_n z^n$ for which $\sum_{n=0}^{\infty}(n+1)^\alpha |a_n| < \infty$ and $\|f(z)\|_\alpha = \sum_{n=0}^{\infty}(n+1)^\alpha |a_n|$, α real.

S_α is an algebra if and only if $\alpha \geqq 0$.

DEFINITION. C is the algebra of functions $f(z)$ continuous in $|z| \leqq 1$ and analytic in $|z|<1$, with $\|f\|_C = \max_{|z|\leqq 1} |f(z)|$. This norm is denoted by $\|f\|_\infty$.

LEMMA 1 (CARATHÉODORY-WALSH). *If $f(z) \in C$ and separates points for $|z| \leqq 1$, then f generates C.*

PROOF. This is an immediate corollary of the theorem in [**3**, p. 36].

LEMMA 2. (1) $\|f\|_\alpha \geqq \min(1, 2^\alpha)\|f'\|_{\alpha-1}$.
(2) $\|f\|_\alpha \leqq \max(1, 2^\alpha)\|f'\|_{\alpha-1}$, *if* $f(0)=0$.

PROOF. $f(z)=\sum_{n=0}^{\infty} a_n z^n$, $f'(z)=\sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}z^n$, $\|f\|_\alpha = \sum_{n=0}^{\infty}(n+1)^\alpha |a_n|$, and $\|f'\|_{\alpha-1} = \sum_{n=0}^{\infty}(n+1)^\alpha |a_{n+1}| = \sum_{n=1}^{\infty} n^\alpha |a_n|$.

$$\|f\|_\alpha = \sum_{n=0}^{\infty}(n+1)^\alpha |a_n| \geqq \sum_{n=1}^{\infty}(n+1)^\alpha |a_n| = \sum_{n=1}^{\infty} n^\alpha \left(\frac{n+1}{n}\right)^\alpha |a_n|$$

$$\geqq \min\left(\frac{n+1}{n}\right)^\alpha \sum_{n=1}^{\infty} n^\alpha |a_n| \geqq \min(1, 2^\alpha)\|f'\|_{\alpha-1}$$

and (1) is proven.

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If $f(0)=a_0=0$ then

$$\begin{aligned}\|f\|_\alpha &= \sum_{n=0}^{\infty}(n+1)^\alpha |a_n| \\ &= \sum_{n=1}^{\infty}(n+1)^\alpha |a_n| = \sum_{n=1}^{\infty} n^\alpha \left(\frac{n+1}{n}\right)^\alpha |a_n| \\ &\leqq \max\left(\frac{n+1}{n}\right)^\alpha \sum_{n=1}^{\infty} n^\alpha |a_n| \leqq \max(1, 2^\alpha)\|f'\|_{\alpha-1}\end{aligned}$$

and (2) is proven.

LEMMA 3. *If $\alpha<0$ then $\|fg\|_\alpha \leqq \|f\|_l \|g\|_\alpha$, where $f\in l$ and $g\in S_\alpha$.*

PROOF. Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$. Then

$$\begin{aligned}\|fg\|_\alpha &= \left\|\left(\sum_{n=0}^{\infty} a_n z^n\right) g\right\|_\alpha \\ &\leqq \sum_{n=0}^{\infty} |a_n| \, \|z^n g\|_\alpha \leqq \left(\sum_{n=0}^{\infty} |a_n|\right) \max\|z^n g\|_\alpha \\ &= \left(\sum_{n=0}^{\infty} |a_n|\right)\|g\|_\alpha = \|f\|_l \|g\|_\alpha.\end{aligned}$$

LEMMA 4. *If $f(z)\in l$ separates points in $|z|\leqq 1$ and $f'(z)$ is continuous in $|z|\leqq 1$ and has no zeroes there, then $f(z)$ generates l.*

PROOF. We note first that $1/f'\in C$ and

$$\begin{aligned}\left\|\sum_{n=0}^{\infty} a_n z^n\right\|_{-1} &= \sum_{n=0}^{\infty}\frac{|a_n|}{n+1} \leqq \left(\sum_{n=0}^{\infty}|a_n|^2\right)^{1/2}\left[\sum_{n=0}^{\infty}\left(\frac{1}{n+1}\right)^2\right]^{1/2} \\ &= \frac{\pi}{\sqrt{6}}\left\|\sum_{n=0}^{\infty} a_n z^n\right\|_{l_2}.\end{aligned}$$

We then have

$$\begin{aligned}\|P(f)-z\|_l &\leqq \|(P(f)-z)'\|_{-1} \qquad \text{(by Lemma 2)} \\ &= \|f'P'(f)-1\|_{-1} \\ &\leqq \frac{\pi}{\sqrt{6}}\|f'P'(f)-1\|_{l_2} \leqq \frac{\pi}{\sqrt{6}}\|f'\|_\infty \left\|P'(f)-\frac{1}{f'}\right\|_\infty\end{aligned}$$

and by Lemma 1 f generates C so $\|P'(f)-1/f'\|_\infty$ can be made arbitrarily small, and f generates l. (The hypothesis on f' can be weakened to $f'\in H^1$. See [1].)

THEOREM 2.1. *$f\in S_\alpha$, $\alpha\geqq 1$, generates S_α if and only if f separates points in $|z|\leqq 1$, and $f'(z)\neq 0$, $|z|\leqq 1$.*

PROOF. If f does not separate points, i.e. there are two points z_1 and z_2 ($z_1\neq z_2$) in the closed disc with $f(z_1)=f(z_2)$, then for any polynomial P we have

$$\begin{aligned}\|P(f)-z\|_\alpha &\geqq \|P(f)-z\|_\infty\\ &\geqq \tfrac{1}{2}(|P(f(z_1))-z_1|+|P(f(z_2))-z_2|)\\ &\geqq \tfrac{1}{2}|P(f(z_1))-z_1+z_2-P(f(z_2))| = \tfrac{1}{2}|z_2-z_1|\end{aligned}$$

so that f does not generate.

If $f'(z_1)=0$, $|z_1|\leqq 1$, then for any polynomial P we have

$$\begin{aligned}\|P(f)-z\|_\alpha &\geqq \|(P(f)-z)'\|_{\alpha-1} \qquad \text{(by Lemma 2)}\\ &\geqq \|(P(f)-z)'\|_l = \|f'P'(f)-1\|_l\\ &\geqq \|f'P'(f)-1\|_\infty \geqq |f'(z_1)P'(f(z_1))-1| = 1\end{aligned}$$

so that f does not generate, and the necessity of the conditions is proven.

Suppose now that f separates points and $f'\neq 0$ in the closed unit disc.

We first show that if f generates S_α and $f\in S_{\alpha+1}$, then f generates $S_{\alpha+1}$.

We note that if $f\in S_{\alpha+1}$ then $f'\in S_\alpha$, and if $f'\neq 0$, $|z|\leqq 1$, then $1/f'\in S_\alpha$ by Wiener's theorem [**2**, p. 299].

If f generates S_α then for any $\epsilon>0$ there is a polynomial Q such that $\|Q(f)-1/f'\|_\alpha<\epsilon/2^{\alpha+1}\|f'\|_\alpha$. Then $\|f'Q(f)-1\|_\alpha\leqq\|f'\|_\alpha\|Q(f)-1/f'\|_\alpha<\epsilon/2^{\alpha+1}$, and $\|P(f)-z\|_{\alpha+1}<\epsilon$, by Lemma 2, where P is that primitive of Q for which $P(f)-z$ has constant term zero. Since $\epsilon>0$ was arbitrary, f generates $S_{\alpha+1}$.

Thus it is sufficient to prove the theorem when $1\leqq\alpha<2$.

For $1\leqq\alpha<2$ we have

$$\begin{aligned}\|P(f)-z\|_\alpha &\leqq 2^\alpha\|P'(f)f'-1\|_{\alpha-1}\leqq 2^\alpha\|f'\|_{\alpha-1}\left\|P'(f)-\frac{1}{f'}\right\|_{\alpha-1}\\ &\leqq 2^{2\alpha-1}\|f'\|_{\alpha-1}\left\|f'P''(f)+\frac{f''}{(f')^2}\right\|_{\alpha-2}.\end{aligned}$$

For any polynomial $Q(z)$ this is in turn

$$\leqq 2^{2\alpha-1}\|f'\|_{\alpha-1}\left[\|f'(P''(f)-Q(z))\|_{\alpha-2}+\left\|f'\left(Q(z)+\frac{f''}{(f')^3}\right)\right\|_{\alpha-2}\right]$$

$$\leqq 2^{2\alpha-1}\|f'\|_{\alpha-1}\left[\|f'(P''(f)-Q(z))\|_{l}+\left\|f'\left(Q(z)+\frac{f''}{(f')^3}\right)\right\|_{\alpha-2}\right]$$

$$\leqq 2^{2\alpha-1}\|f'\|_{\alpha-1}\|f'\|_{l}\left[\|P''(f)-Q(z)\|_{l}+\left\|Q(z)+\frac{f''}{(f')^3}\right\|_{\alpha-2}\right],$$

with Lemma 3 being used in the last inequality. ($f'\in S_{\alpha-1}$ and $f'\neq 0$, $|z|\leqq 1$, so that $1/f'\in S_{\alpha-1}$ by Wiener's theorem, $1/(f')^2\in S_{\alpha-1}$ because $S_{\alpha-1}$ is an algebra, and, differentiating, $f''/(f')^3\in S_{\alpha-2}$.)

Since polynomials are dense in $S_{\alpha-2}$ we can first choose $Q(z)$ so that $\|Q(z)+f''/(f')^3\|_{\alpha-2}$ is arbitrarily small and then, since f generates l, we can choose P'' so that $\|P''(f)-Q(z)\|_l$ is arbitrarily small. Thus f generates S_α, $1\leqq\alpha<2$, and the proof is complete.

References

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