# $R^c$ IS NOT ALMOST LINDELÖF

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ABSTRACT. It is shown that there exists a Baire measure $\mu$ on $R^c$ such that $R^c$ can be covered by elementary open $\mu$-null sets.

A topological space $X$ will be called "almost Lindelöf" if, for every open cover $\mathcal{G}$ of $X$, and every finite Baire measure $\mu$ on $X$, there exist: a countable subcollection $\mathcal{H} \subset \mathcal{G}$, and a $\mu$-null set $E$, such that $E \cup \bigcup \mathcal{H} = X$. From H. Rubin the authors learnt of the following question: Is $R^{\aleph_1}$ almost Lindelöf? We shall show that, on the continuum hypothesis, the answer is "no".

In fact, without hypothesis, we prove:

*There exist, for each cardinal $\mathfrak{M} \geqq c$, a cover $\mathcal{G}$ of $R^{\mathfrak{M}}$ by elementary open sets, and a Baire measure $\mu$ on $R^{\mathfrak{M}}$ with $\mu(R^{\mathfrak{M}}) = 1$, such that each $G \in \mathcal{G}$ is $\mu$-null.*[2]

It is evidently enough to do this when $\mathfrak{M} = c$. Further, to simplify the details, we first work with $N^c$, where $N = \{1, 2, 3, \cdots\}$, rather than $R^c$. It will then be easy to derive an example for $R^c$ from the following one for $N^c$.

Regard $N^c$ as $P \times Q$, where $P = N^{\aleph_0}$ consists of all sequences of positive integers and $Q = N^c$ is regarded as the set $N^P$ of all maps of $P$ into $N$. For each $p \in P$ and $n \in N$, define

$G_n(p) = \{(x, y) \in P \times Q : x_k = p_k \text{ for } k = 1, 2, \cdots, n, \text{ and } y(p) = n\}$.

Then $\mathcal{G} = \{G_n(p) : p \in P, n \in N\}$ is the desired cover of $P \times Q$. It is a cover, because a given $z = (x, y) \in P \times Q$ will belong to $G_n(p)$ where $p = x$ and $n = y(x)$; and it evidently consists of elementary sets.

As a preliminary to defining the desired measure $\mu$, we set up a map $\psi : P \to P \times Q$ as follows. For $x \in P$, define $\psi(x) = (x, y)$ where $y \in Q$ is specified by:

(1) $y(p) = \inf\{n \in N : p_n \neq x_n\} + 1$ if $p \neq x$ $(p \in P)$,

and

(2) $y(x) = 1$.

(Note that $y(p) = 1$ if and only if $p = x$.)

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[2] The construction below is the first author's simplification of the second author's construction.

One easily verifies that $\psi$ is Baire measurable; it is enough to verify that the inverse of each elementary subset of $P\times Q$ belongs to the $\sigma$-field generated by the elementary subsets of $P$.

Now take any Baire measure $\lambda$ on $P=N^{\aleph_0}$, of total measure 1, and such that each singleton has measure 0; an obvious product measure will do this. Define, for each Baire set $B\subset P\times Q$, $\mu(B)=\lambda(\psi^{-1}(B))$. Thus $\mu$ is a Baire measure on $P\times Q$, of total measure 1, and we have only to check that $\mu(G_n(p))=0$ for each $n\in N$ and $p\in P$. To do this, consider $A=\psi^{-1}(G_n(p))$. If $x\in A$ then $\psi(x)=(x, y)\in G_n(p)$, where $y$ is defined by (1) and (2) above. From the definition of $G_n(p)$, we have $x_1=p_1, \cdots, x_n=p_n$, $y(p)=n$. If (1) applies here then $y(p)\geqq n+1$, which is impossible. Hence $y(p)=1$ and $p=x$. In other words, either $A$ is empty or $A$ is the singleton $\{p\}$. In either case, $\lambda(A)=0$; thus $\mu(G_n(p))=0$, and the proof is complete.

Finally, to transfer the example from $N^c$ to $R^c$, we consider $N^c$ embedded as a closed subset of $R^c$ in the obvious way. Note that, for each elementary open set $G$ in $N^c$, there exists an elementary open set $H$ in $R^c$ such that $H\cap N^c=G$. (For instance, if $G$ is $\{z\in N^c: z_\alpha=n_\alpha \text{ for } \alpha=\alpha_1, \alpha_2, \cdots, \alpha_k\}$ we may take $H=\{z\in R^c: |z_\alpha-n_\alpha|<1/2 \text{ for } \alpha=\alpha_1, \alpha_2, \cdots, \alpha_k\}$.) Apply this to obtain, for each $G_n(p)\in\mathcal{G}$, an elementary open set $H_n(p)$ in $R^c$ whose intersection with $N^c$ is $G_n(p)$. Let $\mathcal{H}=\{H_n(p): p\in P, n\in N\}$; this covers $N^c$. Express the open set $R^c-N^c$ as the union of a system $\mathcal{K}$ of elementary open sets in any way. Then $\mathcal{H}\cup\mathcal{K}$ is the desired cover of $R^c$. To obtain the measure on $R^c$, define (for each Baire subset $B$ of $R^c$)

$$\nu(B) = \mu(B\cap N^c).$$

Then $\nu$ is a finite Baire measure on $R^c$, and $\nu(H)=0$ for each $H\in\mathcal{H}\cup\mathcal{K}$.

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