NORMALITY AND PROPERTIES RELATED TO COMPACTNESS IN HYPERSPACES

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Introduction. Let X be a regular T_1 topological space and 2^X the space of all closed nonempty subsets of X with the finite topology [8, Definition 1.7]. In [6] Ivanova has shown that if X is a noncompact ordinal space, then 2^X is nonnormal. In this paper we give a new proof of this fact. This result is then used to show that several properties of 2^X are equivalent to the compactness of X. It is not known if the normality of 2^X is equivalent to the compactness of X. There are some partial results in this direction though. The paracompactness of 2^X is shown to be equivalent to the compactness of X and the normality of 2^{2^X} is also shown to be equivalent to the compactness of X. In the last part of the paper some properties related to the countable compactness of 2^X are investigated.

Notation. Because of our assumptions on X, $\hat{X}=\{\{x\}:x\in X\}$ is a closed subset of 2^X homeomorphic to X. The set $\mathfrak{F}_n(X)=\{F\subset X: F$ has at most n points$\}$ is also closed. Furthermore, the space 2^X is Hausdorff. For notation and further basic results on hyperspaces see [7] or [8]. In particular we use $\langle U_1, \cdots, U_n\rangle=\{A\in 2^X: A\subset \bigcup_{i=1}^n U_i$ and $A\cap U_i\neq\varnothing$ for all $i\}$. If each U_i is open in X, then $\langle U_1, \cdots, U_n\rangle$ is open in 2^X and the set of such sets in 2^X forms a basis for 2^X. By considering such basic open sets it is clear that the set $\mathfrak{F}(X)$ of finite subsets of X is dense in 2^X. We denote the cardinality of a set Z by $|Z|$.

1. **The hyperspace of a discrete space.** It follows from Ivanova's result [6], that the hyperspace of an infinite discrete space is nonnormal. We give a new proof of this result based on the following lemma.

LEMMA. *If X is an infinite discrete space of cardinality σ, then 2^X has a dense subset of cardinality σ and a closed discrete subset of cardinality 2^σ.*

PROOF. The set $\mathfrak{F}(X)$ of finite subsets of X is dense in 2^X with $|\mathfrak{F}(X)|=|X|=\sigma$. To prove the last statement, let $X=X_1\cup X_2$ be a disjoint union with $|X_i|=\sigma$. Let $f_i: X\to X_i$ be a bijection for each i and let $F: 2^X\to 2^X$ be defined by $F(A)=f_1(A)\cup f_2(X-A)$. Let $\mathfrak{A}=\{F(A): A\subset X\}$. Now for $A\subset X$, letting $\mathfrak{U}=2^{F(A)}$ which is open in 2^X, we have $\mathfrak{U}\cap\mathfrak{A}=\{F(A)\}$. Thus $\mathfrak{A}$ is discrete. Now if $B\in 2^X$ and

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$B \notin \mathcal{A}$, then either $f_1^{-1}(B \cap X_1) \cup f_2^{-1}(B \cap X_2) \neq X$ or $f_1^{-1}(B \cap X_1) \cap f_2^{-1}(B \cap X_2) \neq \varnothing$. In either case one can easily show that B is not in the closure of $\mathcal{A}$. Thus $\mathcal{A}$ is closed and discrete. It is clearly of cardinality 2^σ since F is an injection.

THEOREM 1. *If X is infinite and discrete, then 2^X is nonnormal.*

PROOF. Since 2^X has a dense subset of cardinality σ, there are at most 2^σ continuous real valued functions on 2^X. But the existence of $\mathcal{A}$ in the proof of the lemma says that if 2^X is normal, then by extending each characteristic function of each subset of $\mathcal{A}$ to all of 2^X (as real valued functions), there must be at least 2^{2^σ} continuous real valued functions on 2^X. Thus we cannot have 2^X normal.

COROLLARY. *If 2^X is normal, then X is countably compact.*

PROOF. If X is not countably compact, then there is a closed discrete subset D in X. But then 2^D is a closed subspace of 2^X. Since 2^D is nonnormal, 2^X is nonnormal.

2. **Compactness of the hyperspace.** We now show that several properties related to compactness are equivalent to compactness in the hyperspace.

DEFINITION. A topological space is said to be *paracompact* if every open cover has a locally finite refinement [2, p. 162]. It is said to be *metacompact* if each open cover has a point finite open refinement [2, p. 229]. It is *meta-Lindelöf* (countably metacompact in [1]) if each open cover has a point countable open refinement.

LEMMA. *If N is the integers with discrete topology, then 2^N is not meta-Lindelöf.*

PROOF. Since 2^N is separable, if it is meta-Lindelöf it is Lindelöf. However, 2^N is regular [8, Theorem 4.9]. Thus if 2^N is meta-Lindelöf, then 2^N is normal contradicting Theorem 1.

THEOREM 2. *The following are equivalent*:

(a) *X is compact,*

(b) *2^X is compact,*

(c) *2^X is Lindelöf,*

(d) *2^X is paracompact,*

(e) *2^X is metacompact, and*

(f) *2^X is meta-Lindelöf.*

PROOF. It is well known that (a) is equivalent to (b) [8, Theorem 4.2]. It will be sufficient to show that (f) implies (a). If 2^X is meta-

Lindelöf, then so is X since $\hat{X}$ is a closed subset of 2^X. By the corollary to Theorem 1 we must have that X is also countably compact. Thus X is meta-Lindelöf and countably compact, hence compact [1].

3. **Normality of the hyperspace of the hyperspace.** We will now need the fact that if X is a noncompact well ordered space, then 2^X is nonnormal. This result is due to Ivanova [6] and for completeness we now include a proof.

THEOREM 3. *If X is a noncompact ordinal space, then 2^X is nonnormal.*

PROOF. Let $X=[0,\alpha)=\{\gamma:\gamma$ is an ordinal less than $\alpha\}$. By Theorem 1 we need only consider the case that X is countably compact. In this case there is no countable cofinal sequence of ordinals in $[0,\alpha)$. We now give Ivanova's proof for this case. Let $X_\sigma=[\sigma,\alpha)$ for each σ less than α. Then $\mathfrak{F}=\{X_\sigma\}$ is a closed subset of 2^X and $\mathfrak{F}$ and $\hat{X}$ are disjoint in 2^X. If 2^X were normal, there would be an open set $\mathfrak{U}$ containing $\mathfrak{F}$ whose closure misses $\hat{X}$. Let $X_\sigma\in\langle U_1^\sigma,\cdots,U_n^\sigma\rangle\subset\mathfrak{U}$ for each σ where U_i^σ is open in X. Let $\lambda_i^0\in U_i^0\cap X_0$ and $\sigma_1=\max\{\lambda_i^0\}$. Then $\{\lambda_1^0,\cdots,\lambda_n^0\}\in\langle U_1^0,\cdots,U_n^0\rangle$. Now let $\lambda_i^1\in U_i^{\sigma_1}\cap X_{\sigma_1}$ and $\sigma_2=\max\{\lambda_i^1\}$. Then $\{\lambda_1^1,\cdots,\lambda_n^1\}\in\langle U_1^{\sigma_1},\cdots,U_n^{\sigma_1}\rangle$. Continuing this process inductively we get an increasing sequence of ordinals $\{\sigma_j\}$ in $[0,\alpha)$ and a countable collection of finite subsets of X, $\{\lambda_i^j:i=1,\cdots,n_j\}$ with the property that $\{\lambda_i^j\}\in\mathfrak{U}$ for each j and $\{\lambda_i^j:i=1,\cdots,n_j\}\subset[\sigma_j,\sigma_{j+1}]$. Let $\sigma=\sup\{\sigma_j\}$ which is less than α. Then in 2^X the singleton set $\{\sigma\}$ is in the closure of $\mathfrak{U}$ since the sequence $\{\{\lambda_i^j:i=1,\cdots,n_j\}:j=1,2,3,\cdots\}$ converges to $\{\sigma\}$. Thus $\mathfrak{F}$ and $\hat{X}$ cannot be separated in 2^X.

THEOREM 4. *If 2^{2^X} is normal, then X is compact.*

PROOF. We will show that there is a limit ordinal α such that $[0,\alpha)$ can be imbedded as a closed subset of 2^X if X is not compact. The theorem will then follow from Theorem 3 since $2^{[0,\alpha)}$ will then be a closed subspace of 2^{2^X}. Now suppose that X is not compact and let $\mathfrak{F}$ be a filter base of closed sets with empty intersection. Let us suppose that $|\mathfrak{F}|=\sigma$ and that σ is the minimum cardinal with the property that there is such a filter base. That is, if $\mathfrak{G}$ is a collection of closed sets in X with finite intersection property with $|\mathfrak{G}|$ less than σ, then the intersection of $\mathfrak{G}$ is nonempty. Identify σ with the first ordinal having that cardinality. Then $[0,\sigma)$ has cardinality σ and $[0,\gamma)$ has cardinality less than σ for γ less than σ. Then let $\mathfrak{F}$ be indexed by $\{F_\alpha:\alpha<\sigma\}$. By transfinite induction we define a sequence $\{\alpha_\gamma:\gamma<\sigma\}$

in $[0, \sigma)$ in the following manner. We let $\alpha_0 = 0$. Having defined α_γ let $\alpha_{\gamma+1}$ be the first α such that $\bigcap \{ F_\alpha : \alpha \leqq \alpha_{\gamma+1} \}$ is properly contained in $\bigcap \{ F_\alpha : \alpha \leqq \alpha_\gamma \}$. For γ a limit ordinal, let $\alpha_\gamma = \sup \{ \alpha_\lambda : \lambda < \gamma \}$. It is clear that this process may be continued until $\bigcap \{ F_{\alpha_\gamma} \} = \emptyset$. But this will require using all γ's less than σ by the minimality of σ. Now let $G_\gamma = \bigcap \{ F_\alpha : \alpha \leqq \alpha_\gamma \}$. Then G_γ is closed and $G_\gamma \neq \emptyset$ for all γ. Thus $G_\gamma \in 2^X$. We now show that $\{ G_\gamma \}$ is our required set.

Claim. The set $\{ G_\gamma : \gamma < \sigma \}$ is a closed subset of 2^X.

Proof of claim. If F is an element of 2^X and not of $\{ G_\gamma \}$, then if F is not contained in any G_γ let $\mathfrak{U} = 2^X - 2^{G_0}$. Then $\mathfrak{U}$ is an open set in 2^X containing F and containing no element of $\{ G_\gamma \}$. Now if $F \subset G_\gamma$ for some γ, then let $A = \bigcap \{ G_\gamma : F \subset G_\gamma \}$. Then $A = G_\lambda$ where $\lambda = \sup \{ \gamma : F \subset G_\gamma \}$ and F is not contained in $G_{\lambda+1}$. Let $x \in G_\lambda - F$ and let $\mathfrak{U} = 2^{X - \{x\}} - 2^{G_{\lambda+1}}$. Then $F \in \mathfrak{U}$ with $\mathfrak{U}$ open and $\mathfrak{U} \cap \{ G_\gamma \} = \emptyset$.

Claim. If $f(\gamma) = G_\gamma$, then f is a homeomorphism from $[0, \sigma)$ onto $\{ G_\gamma \}$.

Proof of claim. Clearly f is one to one and onto. We now show that f is bicontinuous. Let $G_\lambda \in \{ G_\gamma \}$ and $x \in G_\alpha - G_{\lambda+1}$ for any $\alpha < \lambda$. Then if $\mathfrak{U} = 2^{X - \{x\}} - 2^{G_{\lambda+1}}$, then $f([\alpha+1, \lambda]) = \mathfrak{U} \cap \{ G_\gamma \}$. Thus f^{-1} is continuous. Now let $\lambda \in [0, \sigma)$ and let $G_\lambda \in \langle U_1, \cdots, U_n \rangle$. Since $G_{\lambda+1}$ is closed and properly contained in G_λ we may assume that $\langle U_1, \cdots, U_n \rangle \cap \{ G_\gamma \} = \{ G_\gamma : G_\gamma \supset G_\lambda$ and $G_\gamma \subset \bigcup_{i=1}^n U_i \}$ as a typical basic neighborhood of G_λ in $\{ G_\gamma \}$. Now suppose that the set $\{ \gamma : G_\gamma \subset \bigcup_{i=1}^n U_i$ and $G_\gamma \supset G_\lambda \}$ does not contain λ in its interior. Then there is a set $\{ \gamma_\alpha \}$ such that (1) $\gamma_\alpha < \lambda$ for all α, (2) $\sup \{ \gamma_\alpha \} = \lambda$, and (3) G_{γ_α} is not contained in $\bigcup_{i=1}^n U_i$ for each α. Now $\{ G_{\gamma_\alpha} - \bigcup_{i=1}^n U_i \}$ has the finite intersection property and empty intersection. Thus $\{ \gamma_\alpha \}$ has cardinality σ and $\{ \gamma_\alpha \}$ is cofinal in $[0, \sigma)$, contradicting the fact that $\sup \{ \gamma_\alpha \} = \lambda < \sigma$. Thus λ is in the interior of $\{ \gamma : G_\gamma \subset \bigcup_{i=1}^n U_i$ and $G_\gamma \supset G_\lambda \}$ and f is continuous. Thus f is a homeomorphism. The proof of Theorem 4 is now complete.

4. Countable compactness and hyperspaces.

We show that 2^X can be countably compact and noncompact. We also give an example of a countably compact completely regular space X such that 2^X is not pseudocompact.

Definition. A space X is *pseudocompact* if each real valued continuous function is bounded. A space is *strongly countably compact* if each countable set has compact closure.

If a space is strongly countably compact, then it is countably compact. If it is countably compact, then it is pseudocompact. Neither of these implications is reversible in general. The property of being strongly countably compact has very interesting properties.

It is productive, closed hereditary, and preserved under continuous transformation. The next theorem gives an additional result for this property. The following lemma is well known [**9**]. (Compare [**5**, 6.5 (IV), p. 86].)

LEMMA. *If X is normal and $\{F_1, \cdots, F_k\}$ is a finite collection of closed subsets of X, then* $\bigcap_{i=1}^k \mathrm{Cl}_{\beta X} F_i = \mathrm{Cl}_{\beta X}[\bigcap_{i=1}^k F_i]$.

PROOF. We will show this for two closed sets F_1 and F_2 in X. The lemma will then follow by induction. Clearly, $\mathrm{Cl}_{\beta X}(F_1 \cap F_2) \subset \mathrm{Cl}_{\beta X} F_1 \cap \mathrm{Cl}_{\beta X} F_2$. Suppose that x is an element of $\mathrm{Cl}_{\beta X} F_1 \cap \mathrm{Cl}_{\beta X} F_2$ and not of $\mathrm{Cl}_{\beta X}(F_1 \cap F_2)$. Let $x \in V$ with V open in βX such that $\mathrm{Cl}_{\beta X} V \cap \mathrm{Cl}_{\beta X}(F_1 \cap F_2) = \varnothing$. Then let $A_i = F_i \cap \mathrm{Cl}_X V$ for $i=1$ and $i=2$. Then x is an element of $\mathrm{Cl}_{\beta X} A_1 \cap \mathrm{Cl}_{\beta X} A_2$ since $x \in \mathrm{Cl}_{\beta X} F_i$ for each i. But $A_1 \cap A_2 = \varnothing$ since $\mathrm{Cl}_X V \cap (F_1 \cap F_2) = \varnothing$. By the normality of X we have $\mathrm{Cl}_{\beta X} A_1 \cap \mathrm{Cl}_{\beta X} A_2 = \varnothing$, a contradiction. Thus $\mathrm{Cl}_{\beta X}(F_1 \cap F_2) = \mathrm{Cl}_{\beta X} F_1 \cap \mathrm{Cl}_{\beta X} F_2$.

LEMMA. *If X is normal and $F: 2^X \to 2^{\beta X}$ is defined by $F(K) = \mathrm{Cl}_{\beta X} K$, then F is an imbedding.*

PROOF. Suppose that $F(K) \in \langle U_1, \cdots, U_n \rangle$ for $K \in 2^X$, with each U_i open in βX. Let $\{V_1, \cdots, V_n\}$ be a collection of open subsets of βX with $F(K) \subset \bigcup_{i=1}^n V_i$, $\mathrm{Cl}_X V_i \subset U_i$ for each i, and with $F(K) \cap V_i \neq \varnothing$ for each i. Let $O_i = V_i \cap X$. One can easily verify that $K \in \langle O_1, \cdots, O_n \rangle$ and that, for $L \in 2^X$ with $L \in \langle O_1, \cdots, O_n \rangle$, $F(L) \in \langle U_1, \cdots, U_n \rangle$.

Now suppose that $K \in 2^X$ with $K \in \langle U_1, \cdots, U_n \rangle$. Then, of course, $K \subset \bigcup_{i=1}^n U_i$. Thus $K \cap (\bigcap_{i=1}^n (X - U_i)) = \varnothing$. By the foregoing lemma, $\mathrm{Cl}_{\beta X} K \cap (\bigcap_{i=1}^n (\mathrm{Cl}_{\beta X}(X - U_i))) = \varnothing$. If we let $O_i = \beta X - \mathrm{Cl}_{\beta X}(X - U_i)$, then the last statement implies that $\mathrm{Cl}_{\beta X} K \subset \bigcup_{i=1}^n O_i$. Thus $F(K) \in \langle O_1, \cdots, O_n \rangle$. But if $F(L) \in \langle O_1, \cdots, O_n \rangle$, then $L \in \langle U_1, \cdots, U_n \rangle$. Thus F^{-1} is continuous from $F(2^X)$ onto 2^X, and thus F is an imbedding.

THEOREM 5. *If X is normal and strongly countably compact, then 2^X is strongly countably compact.*

PROOF. Let $F: 2^X \to 2^{\beta X}$ be defined as in the previous lemma. Then, by the previous lemma, F is an imbedding. Let $\{K_i\}_{i=1}^\infty$ be a countable subset of 2^X. Let $\mathcal{B}$ be the closure of $\{F(K_i)\}_{i=1}^\infty$ in $2^{\beta X}$ and $\mathcal{C}$ the closure of $\{K_i\}_{i=1}^\infty$ in 2^X. Since F is an imbedding, $F(\mathcal{C}) = \mathcal{B} \cap F(2^X)$. If we can show that $\mathcal{B} \subset F(2^X)$, then $F(\mathcal{C}) = \mathcal{B}$ and thus $\mathcal{C}$ will be compact since $\mathcal{B}$ is. Thus it will be sufficient to prove that $\mathcal{B} \subset F(2^X)$.

Let K^* be any element of $\mathfrak{B}$ in $2^{\beta X}$, and let $\mathrm{Cl}_{\beta X}K_{i_\alpha}$ be a net converging to K^* in $2^{\beta X}$. Let $K=K^*\cap X$. Now suppose that $x\in K^*-\mathrm{Cl}_{\beta X}K$. Then let $x\in U$ with U open in βX with $[\mathrm{Cl}_{\beta X}U]\cap[\mathrm{Cl}_{\beta X}K]=\varnothing$. Since $\mathrm{Cl}_{\beta X}K_{i_\alpha}\to K^*$, there must be a γ such that, for $\alpha\geqq\gamma$, $[\mathrm{Cl}_{\beta X}K_{i_\alpha}]\cap U\neq\varnothing$. Thus $K_{i_\alpha}\cap U\neq\varnothing$ for $\alpha\geqq\gamma$. Let $A=\{i:K_i\cap U\neq\varnothing\}$. Let $a_i\in K_i\cap U$ for each $i\in A$. Let $B=\mathrm{Cl}_X\{a_i:i\in A\}$. Then B is compact by the strong countable compactness of X. Therefore there is a subnet of a_{i_α} converging to some $a\in B$. One can easily show that $a\in K^*$ and thus $a\in K^*\cap X=K$. But also $a\in\mathrm{Cl}_{\beta X}U$ and $[\mathrm{Cl}_{\beta X}U]\cap K\neq\varnothing$, a contradiction. Therefore we must have that $K^*=\mathrm{Cl}_{\beta X}K$. Thus $K^*\in F(2^X)$ and $\mathfrak{B}\subset F(2^X)$.

It would be interesting to know if in Theorem 5 the assumption of normality can be reduced. At any rate, the theorem is sufficient for the next application.

Example. Let X be a noncompact countably compact ordinal space. Then X is strongly countably compact and normal. Thus 2^X is strongly countably compact. Thus we have an example of a noncompact space X for which 2^X is countably compact and hence pseudocompact.

Example. For each integer n greater than 1, Frolík [3] has given an example of a space X such that X^{n-1} is countably compact and X^n is not pseudocompact. In the examples X^n fails to be pseudocompact by containing an open and closed copy of the integers N. Let $p:X^n\to\mathfrak{F}_n(X)$ be defined by $p((x_1,\cdots,x_n))=\{x_1,\cdots,x_n\}$. Then p is closed and continuous [4]. Now let N be an open and closed copy of the integers in X^n. Then $p(N)$ is closed in $\mathfrak{F}_n(X)$ and thus is closed in 2^X. Now for $(x_1,\cdots,x_n)$ an isolated point in X^n, we have $\{x_1,\cdots,x_n\}$ is also an isolated point in 2^X. Thus $p(N)$ is open and closed in 2^X. Since p is finite to one, $p(N)$ is infinite and 2^X contains an open and closed copy of the integers. Thus 2^X is nonpseudocompact. Thus Frolík's examples give examples of countably compact completely regular spaces whose hyperspaces are not pseudocompact.

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Added in proof. Assuming the continuum hypothesis the author has been able to show that 2^X is normal if and only if X is compact.

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