# EXPANSIVE HOMEOMORPHISMS ON COMPACT MANIFOLDS

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ABSTRACT. In this paper theorems are proved which provide for lifting and projecting expansive homeomorphisms through pseudo-covering mappings so that the lift or projection is also an expansive homeomorphism. Using these techniques it is shown that the compact orientable surface of genus 2 admits an expansive homeomorphism.

1. **Introduction.** Given a light open map $\phi: X\to Y$, the branch set $B_\phi$ is the set at which $\phi$ fails to be a local homeomorphism. If the restriction of $\phi$ to $X-\phi^{-1}(\phi(B_\phi))$ is a finite-to-one covering map then $\phi$ will be called a *pseudo-covering map* (see Definition 5 in [2]). A homeomorphism, $f$, of a space $X$ with metric $d$, will be called *expansive* (*with expansive constant* $c>0$) if to each pair of distinct points $x$, $y$ of $X$ there corresponds an integer $n$ such that $d(f^n(x), f^n(y))>c$.

In [4] theorems are proved for lifting and projecting expansive homeomorphisms through covering maps and for lifting them through pseudo-covering maps. In §2 of this paper a theorem for projecting expansive homeomorphisms through pseudo-covering maps is proved. In §3, Corollary 3.2 provides for the existence of expansive homeomorphisms on 2-manifolds with less stringent conditions than those of Corollary 4.6 of [4]. In §4 it is proved that the compact orientable surface of genus 2 admits an expansive homeomorphism.

2. **A projection theorem.** In the following $M$ and $N$ are compact $n$-manifolds with metrics $r$ and $d$ respectively. Let $\phi: M\to N$ be a pseudo-covering map with $\phi$ one-to-one on $\phi^{-1}(\phi(B_\phi))$. In particular $\phi^{-1}(\phi(B_\phi))=B_\phi$ and since $B_\phi$ is closed the restriction of $\phi$ to $B_\phi$ is a homeomorphism. Let $f: M\to M$ be a fibre-preserving homeomorphism such that $f$ sends $B_\phi$ onto $B_\phi$. Thus $f$ induces a homeomorphism $g: N\to N$.

DEFINITION 2.1. The map $f$ is *expansive on fibres* if there is a number $e>0$ such that for any two points $x$, $y$ in $N$ there exists an integer $n$ with min $r(f^n(z), f^n(w))>e$ for each pair $z\in\phi^{-1}(x)$, $w\in\phi^{-1}(y)$.

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THEOREM 2.2. *Let $\phi: M \to N$ be a pseudo-covering map and $f: M \to M$ a fibre-preserving homeomorphism with $f: B_\phi \to B_\phi$. Let $g: N \to N$ be the homeomorphism induced on $N$. Then $g$ is expansive if and only if $f$ is expansive on fibres.*

PROOF. Suppose $f$ is expansive on fibres with corresponding constant $e$. Let $x$, $y$ be in $N$. We consider three cases.

*Case* 1. Both $x$ and $y$ are in $\phi(B_\phi)$. We have assumed that $\phi$ is one-to-one on $B_\phi$. Thus the restriction of $g$ to $\phi(B_\phi)$ can be expressed as $\phi f \phi^{-1}$. Since $\phi(B_\phi)$ is compact $\phi^{-1}$ is uniformly continuous on $\phi(B_\phi)$ and thus Bryant's theorem [**1**, Theorem 1] applies to prove $g$ expansive on $\phi(B_\phi)$ with some expansive constant $c$.

*Case* 2. Both $x$ and $y$ in $N - \phi(B_\phi)$. For each $b \in B_\phi$ choose an open neighborhood $S_b$ with center $b$ such that diameter $S_b < e/3$ and also such that if $r(a, b) > e/3$ then $\phi(S_a) \cap \phi(S_b)$ is empty. This can be done since the restriction of $\phi$ to $B_\phi$ is a homeomorphism. Now for each $x \in N - \phi(B_\phi)$ choose an elementary neighborhood $V_x$ for the covering map associated with $\phi$ such that the components $U_{x,i}$ of $\phi^{-1}(V_x)$ have diameter $< e$. The sets $\{\phi(S_b)\} \cup \{V_x\}$ form an open cover of $N$. Let $\beta$ be the Lebesgue number of this cover. Since $f$ is expansive on fibres there is an $m$ such that the points $f^m(z)$, $f^m(w)$ are not both contained in any of the sets $U_{x,i}$ or $S_b$ where $z$ is any point of the fibre over $x$ and $w$ is any point of the fibre over $y$. Also if $f^m(z)$ is in $S_a$ and $f^m(w)$ is in $S_b$ then $r(a, b) > e/3$ and thus $\phi(S_a) \cap \phi(S_b)$ is empty. It follows that the points $g^m(x)$, $g^m(y)$ are not both contained in any element of the cover for $N$ and therefore $d(g^m(x), g^m(y)) > \beta$.

*Case* 3. $x \in \phi(B_\phi)$ and $y \in N - \phi(B_\phi)$. Let $\eta$ be greater than 0 and consider the open set $N(B_\phi, \eta)$ of points whose distance from $B_\phi$ is less than $\eta$. The function $T: B_\phi \times (M - N(B_\phi, \eta)) \to R$ defined by $T(b, m) = d(\phi(b), \phi(m))$ is never zero and hence is always greater than some positive number, $h$. Thus if the fibre over $g^m(y)$ lies outside of $N(B_\phi, \eta)$ for some $m$ then $d(g^m(x), g^m(y)) > h$. We consider what happens when the fibre over $y$ remains close to $B_\phi$. Since the restriction of $\phi$ to $B_\phi$ is a homeomorphism there is a $\delta > 0$ such that if $a$, $b$ are in $B_\phi$ and $d(\phi(a), \phi(b)) < \delta$ then $r(a, b) < e/2$. Since $\phi$ is uniformly continuous there is a $\mu > 0$ such that $r(x, y) < \mu$ implies $d(\phi(x), \phi(y)) < \delta/2$. Choose $\eta < \min(e/2, \mu)$. Let $b = \phi^{-1}(x)$. Since $f$ is expansive on fibres we can choose $m$ so that $r(f^m(b), f^m(w)) > e$ for each $w$ in $\phi^{-1}(y)$. Suppose $f^m(w)$ is in $N(B_\phi, \eta)$. Then $r(a, f^m(w)) < \eta$ for some $a \in B_\phi$ and $r(f^m(b), a) > e/2$. Then $d(\phi f^m(w), \phi(a)) = d(g^m(y), \phi(a)) < \delta/2$ and $d(\phi f^m(b), \phi(a)) = d(g^m(x), \phi(a)) > \delta$ and hence $d(g^m(x), g^m(y)) > \delta/2$. Thus $g$ is expansive with expansive constant $\min(c, \beta, h, \delta/2)$.

Conversely suppose $f$ is not expansive on fibres. Let $c$ be any constant $>0$. Choose $e>0$ such that $r(u, v)<e$ implies $d(\phi(u), \phi(v))<c$. There exist points $x$, $y$ in $N$ such that for any $m$ there is a $z\in\phi^{-1}(x)$ and a $w\in\phi^{-1}(y)$ and $r(f^m(z), f^m(w))<e$. Therefore $d(g^m(x), g^m(y))<c$ for all $m$ and $g$ is not expansive.

The following theorem, analogous to Theorem 1 in [1], may be useful for modifying homeomorphisms expansive on fibres. Its proof follows that of Bryant's theorem.

THEOREM 2.3. *If $\phi: M\to N$ is a pseudo-covering map, $g: M\to M$ is a fibre-preserving homeomorphism, and $f: M\to M$ is a homeomorphism which is expansive on fibres, then $g^{-1}fg$ is expansive on fibres.*

PROOF. Let $f$ be expansive on fibres with expansive constant $e$. Using the uniform continuity of $g$, choose $\eta$ so that $r(a, b)<\eta$ implies $r(g(a), g(b))<e$. Now given $x$, $y$ in $N$ the sets $\{g(z): z\in\phi^{-1}(x)\}$ and $\{g(w): w\in\phi^{-1}(y)\}$ are fibres. Thus there is an $m$ such that $r(f^m g(z), f^m g(w))>e$ for each $z$ and $w$. It follows that

$$r((g^{-1}fg)^m(z), (g^{-1}fg)^m(w)) = r((g^{-1}f^m g(z), g^{-1}f^m g(w))>\eta$$

for each $z$ and $w$ and thus $g^{-1}fg$ is expansive on fibres with expansive constant $\eta$.

REMARK 2.4. The three-fold iterate of Reddy's torus homeomorphism [5, p. 631] is fibre preserving with respect to a pseudo-covering map, $\phi$ of $S^2$ by the torus. The map $\phi$ is defined by $\phi(x, y, z) = (x^2-y^2, 2xy, z)$ with the torus represented as in §4 below. However this homeomorphism is not expansive on fibres and hence the homeomorphism induced on $S^2$ is not expansive.

3. **Lifting expansive homeomorphisms.** Again, $\phi: M\to N$ will be a pseudo-covering mapping of a compact $n$-manifold with $\phi$ one-to-one on $\phi^{-1}(\phi(B_\phi))$. We will also use $\phi$ for the associated covering map $\phi: M-B_\phi\to N-\phi(B_\phi)$.

THEOREM 3.1. *Let $g$: $N\to N$ be a homeomorphism which sends $\phi(B_\phi)$ onto $\phi(B_\phi)$. Assume that dimension $B_\phi<n-1$. Then $g$ lifts to a homeomorphism $f: M\to M$ if and only if $g_*\phi_*\pi_1(M-B_\phi, m)$ is conjugate to $\phi_*\pi_1(M-B_\phi, m')$ in $\pi_1(N-\phi(B_\phi), g(n))$. Here $m$ is a point in $\phi^{-1}(n)$ and $m'$ is in $\phi^{-1}(g(n))$.*

PROOF. The condition of the theorem is equivalent to the statement that the pairs $(M-B_\phi, \phi)$ and $(M-B_\phi, g\phi)$ are isomorphic covering spaces of $N-\phi(B_\phi)$. Hence the condition implies that $g$ lifts to a homeomorphism $f: M-B_\phi\to M-B_\phi$ and, by the extension theorem of Fox [3, p. 247], $f$ extends uniquely to a homeomorphism $f: M\to M$ which covers $g$.

In particular, if $g$ is homotopic to the identity by a homotopy which leaves $\phi(B_\phi)$ fixed, then the condition of this theorem is satisfied. This yields the following improvement of Corollary 4.6 in [4].

COROLLARY 3.2. *If $g$ is an expansive homeomorphism of the 2-manifold $N$ on itself, $\phi$ is a pseudo-covering map of $M$ onto $N$ and $g$ is homotopic to the identity by a homotopy which leaves the set $\phi(B_\phi)$ pointwise fixed, then $g$ can be lifted to an expansive homeomorphism of $M$.*

PROOF. Since $M$ and $N$ are compact 2-manifolds, $B_\phi$ consists of a finite set of points and thus has dimension 0. The proof then follows from the above observation on Theorem 3.1, and an application of Theorem 3.4 in [4].

4. **Expansive homeomorphism for a surface of genus** 2. In this section we prove the following.

THEOREM 4.1. *The compact orientable surface of genus 2 admits an expansive homeomorphism.*

PROOF. Let $M$ be the surface in question and $N$ the torus. The proof consists in showing that if $g$ is the three-fold iterate of the Reddy torus homeomorphism, then $g$ lifts to $M$ through the pseudo-covering map of $N$ by $M$ described in [4, §5]. Then Theorem 3.4 of [4] completes the proof. It suffices to show that $g$ lifts through the associated covering map.

Represent the torus as the set of points,

$$\{(x, y, z) \in R^3 : ((x^2 + z^2)^{1/2} - 2)^2 + y^2 = 1\}.$$

The branch set image then consists of the two points $(0, 0, 3)$ and $(0, 0, -3)$. Choose $(0, 0, 1)$ as basepoint, $m$. Then $\pi_1(M-\phi(B_\phi), m)$ is a free group on three generators $\alpha$, $\beta$, and $\gamma$ where $\alpha$ is a loop around the torus in the portion $x>0$, $\beta$ corresponds to the circle $x^2+z^2=1$, and $\gamma$ is represented by a small circle around the point $(0, 0, 3)$. The homeomorphism $g$ is induced by a linear map of the plane. We identify the lower half of the unit square with the back half of the torus, the portion corresponding to $y<0$. Thus $(0, 1/2)$, $(1/2, 1/2)$ correspond to the branch point images $(0, 0, 3)$, $(0, 0, -3)$ respectively. The homeomorphism $g$ leaves the points on the $z$-axis fixed. The image of the fundamental group of $M-B_\phi$ under $\phi_*$ is the subgroup of index 2 generated by $\alpha$, $\beta$, $\gamma\alpha\gamma^{-1}$, $\gamma\beta\gamma^{-1}$, and $\gamma^2$. This set of generators corresponds to the Schreier system consisting of 1 and $\gamma$. Under $g_*$ we have the following:

$$\begin{aligned} g_*(\gamma) &= \gamma \\ g_*(\alpha) &= \alpha\beta\gamma\alpha\beta\gamma\alpha^2\beta\gamma\alpha^2\beta\alpha\beta\gamma\alpha^2\beta\gamma\alpha^2\beta\alpha\beta\gamma\alpha, \\ g_*(\beta) &= \alpha\beta\gamma\alpha\beta\gamma\alpha^2\beta\gamma\alpha^2\beta\alpha\beta\gamma\alpha, \\ g_*(\gamma\alpha\gamma^{-1}) &= \gamma g_*(\alpha)\gamma^{-1}, \\ g_*(\gamma\beta\gamma) &= \gamma g_*(\beta)\gamma^{-1}, \\ g_*(\gamma^2) &= \gamma^2. \end{aligned}$$

It is easy to check that under $g_*$ the generators of the subgroup go into the subgroup and thus $g_*$ sends this subgroup onto itself. Therefore $g$ lifts through the associated covering map and we are finished.

REMARK 4.2. If we consider the pseudo-coverings of the torus by surfaces of higher genus this situation does not occur. However it may be possible to show that under $g_*$ the subgroup of $\pi_1(N-\phi B_\phi)$ corresponding to the associated covering map goes into a conjugate subgroup and thus get a lifting.

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