A CLASS OF NON-NOETHERIAN DOMAINS

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ABSTRACT. A new class of non-noetherian domains, called β-domains, are characterized in the first part of this paper. The second part is concerned with deciding when the intersection of a β-domain with a valuation ring is again a β-domain.

1. **Introduction.** Let k be a field and v a nondiscrete valuation defined on k. All rings under consideration will be subdomains of k and will contain the multiplicative identity of k. Denote the valuation ring of v by R_v and its maximal ideal by M_v. A subdomain R of R_v is called a *β-domain* for v in case R contains a sequence $\{a_i\}_{i=1}^{\infty}$ with the property that $v(a_1)>v(a_2)>v(a_3)\cdots$. $\{a_i\}$ is called a *β-sequence for v in R*. In §2 of this paper elementary properties of β-domains and β-sequences are given, β-domains are characterized (Theorems 1 and 2), and sufficient conditions are given for a ring to be a β-domain (Proposition 1).

In general, given two subdomains of a field k it is not possible to determine whether their intersection is a noetherian domain or a non-noetherian domain. Recently Heinzer and Ohm, [5], have announced some results concerning the intersection of noetherian rings and valuation rings. In particular they have shown that if D and R are domains with the same quotient field k, and V is a rank one valuation ring of k such that $R\not\subset V$ and $D=R\cap V$, then:

(i) if V is centered on a finitely generated ideal of D, then V is noetherian; and

(ii) if V is centered on a maximal ideal of D, then D is noetherian if and only if R and V are noetherian.

In §3 we prove theorems related to this result. Theorem 3 analyzes what happens when we consider the intersection of a β-domain R, for a rank one valuation v, with an arbitrary valuation ring. Theorem 4 is a version of Theorem 3 when v is assumed to be of rank $m>1$.

2. **β-domains and β-sequences.** β-domains can be characterized very easily as follows:

Received by the editors July 30, 1969 and, in revised form, September 15, 1969.

AMS Subject Classifications. Primary 1315, 1398.

Key Words and Phrases. Valuation ring, nondiscrete valuations, value group, maximal ideal.

[1] This research was partially supported by NSF Grant GP 11891.

THEOREM 1. *Let R be a subdomain of k. R is a β-domain for some valuation v if and only if R contains a sequence $\{a_i\}_{i=1}^{\infty}$ such that $\{a_i/a_{i+1}\}_{i=1}^{\infty}$ generates a proper ideal in the ring $T=R[a_1/a_2, a_2/a_3, \cdots]$.*

PROOF. Suppose that R is a β-domain for v, then R contains a sequence $\{a_i\}$ such that $v(a_i)>v(a_{i+1})>0$ for all i. Thus $\{a_i/a_{i+1}\} \subset M_v \cap T \neq T$, where $T=R[a_1/a_2, a_2/a_3, \cdots]$. Conversely, let P be a prime ideal in T containing $\{a_i/a_{i+1}\}$. If (R_v, M_v) is a valuation ring in k with center P in T, then $0<v(a_i/a_{i+1})=v(a_i)-v(a_{i+1})$ for each i. Q.E.D.

We now list, without proof, some elementary properties of β-domains and the ring T.

(1) R_v is a β-domain for v.

(2) All β-domains are non-noetherian.

(3) Any ring between a β-domain and R_v is also a β-domain.

(4) Any infinite subsequence of a β-sequence is still a β-sequence.

Let T be the ring which appeared in Theorem 1 and suppose that A is the proper ideal of T generated by $\{a_i/a_{i+1}\}_{i=1}^{\infty}$.

(5) Let t be a fixed positive integer. If $i<t$, then $a_i/a_t \in A$. If $i>t$, then $a_i/a_t \notin T$. For all i, $1/a_i \notin T$.

(6) For each i, a_{i+1} properly divides a_i in T. Hence, the ideal in T generated by $a_1, a_2, \cdots, a_i$ is principal and is in fact generated by a_i.

PROPOSITION 1. (i) *If R is a domain which contains a prime ideal P such that R_P is a nondiscrete valuation ring, then R is a β-domain.*

(ii) *If v is a nondiscrete valuation on a field k and if $R_v=R_1, R_2, \cdots, R_n$ are valuation rings with quotient field k, then $R=\bigcap_{i=1}^{n} R_i$ is a β-domain.*

(iii) *If F is a field, and if $\{X_i\}_{i=1}^{\infty}$ is a set of indeterminates, then $R=F[X_1, X_2, \cdots]$ is a β-domain.*

PROOF. (i) Choose a β-sequence $\{a_i\}$ in $R_P=R_v$. Write $a_i=b_i/c_i$ where $b_i, c_i \in R$ and $c_i \notin P$. Then $v(a_i)=v(b_i)$ for all i, which implies that $\{b_i\}$ is a β-sequence for v in R.

(ii) Let $P=M_v \cap R$. By [**3**, Chapter 6, p. 132], we have $R_P=R_v$. Now apply (i).

(iii) The proof of this statement is contained in the following more general lemma.

First we make the following definition. A finite subset $a_1, \cdots, a_n \in R$ which generates a proper ideal A of R is said to be *analytically independent* in case the following property holds: If $f(Z_1, \cdots, Z_n)$ is a form of arbitrary degree in $R[Z_1, \cdots, Z_n]$ such that $f(a_1, \cdots, a_n)=0$, then all the coefficients of f are in the radical of A. An infinite subset $\{a_i\}_{i=1}^{\infty}$ in R is analytically independent if $\{a_i\}$ generates a proper ideal in R and if each finite subset of $\{a_i\}$ is analytically independent. Note in particular that the set $\{X_i\}$ in part (iii) of Proposition 1, is analytically independent.

LEMMA. *If R contains an infinite analytically independent set, then R is a β-domain for an appropriate v.*

PROOF. Let $\{a_i\}$ be an analytically independent subset of R and let $T=R[a_1/a_2, a_2/a_3, \cdots]$. In view of Theorem 1, it is sufficient to show that the ideal generated by $a_1/a_2, a_2/a_3, \cdots$ is a proper ideal in T. Suppose not, then

$$1 = \sum_{i=1}^{r} \alpha_i (a_1/a_2)^{t_{1i}} (a_2/a_3)^{t_{2i}} \cdots (a_n/a_{n+1})^{t_{ni}}$$

where $\alpha_i \in R$ and t_{ij} are nonnegative integers. Let $t_j = \max_{i=1}^{r} \{t_{ji}\}$. Then

$$(*) \quad 0 = -a_2^{t_1} a_3^{t_2} \cdots a_{n+1}^{t_n} + \sum_{i=1}^{r} \alpha_i a_2^{t_1 - t_{1i}} \cdots a_{n+1}^{t_n - t_{ni}} a_1^{t_{1i}} \cdots a_n^{t_{ni}}.$$

Each term of (*) has degree $\sum_{j=1}^{n} t_j$. Since $\{a_i\}$ is analytically independent, -1 is in the radical of A where $A=(a_1, \cdots, a_n)$. Hence $A=R$, a contradiction. This proves the lemma and also Proposition 1 (iii).

Many non-noetherian rings contain a sequence $\{a_i\}_{i=1}^{\infty}$ such that a_{i+1} properly divides a_i for all i, (e.g., the ring T of Theorem 1 or any nondiscrete valuation ring). In the next theorem we characterize β-domains which have this property.

THEOREM 2. *Let R be a domain and let $\{a_i\}_{i=1}^{\infty}$ be a sequence in R with the property that each a_{i+1} properly divides a_i in R. Then $c_i a_{i+1} = a_i$ where $c_i \in R$. A subsequence of $\{a_i\}$ is a β-sequence in R for an appropriate valuation v if and only if there exists a maximal ideal M of R containing infinitely many c_i.*

PROOF. Suppose that $\{c_i\}$ contains an infinite subsequence whose members lie in M, where M is a maximal ideal of R. We define a subsequence of $\{a_i\}$ inductively. Let $b_1=a_1$ and assume that b_j has been defined. Then $b_j=a_n$ for some n. Choose $m>n$ such that $c_{m-1}\in M$ and define $b_{j+1}=a_m$. Then $b_j=a_n=c_n \cdots c_{m-1}a_m = c_n \cdots c_{m-1}b_{j+1}$, which implies that $b_j/b_{j+1}\in M$. This is true for all j. Hence $\{b_j\}$ is a β-sequence for any valuation v having center M on R.

Conversely, suppose that $\{b_j\}$ is a β-subsequence of $\{a_i\}$. Then each b_{j+1} properly divides b_j, and by Theorem 1 $\{b_j/b_{j+1}\}$ is a subset of some maximal ideal M of the ring $T=R[b_1/b_2, b_2/b_3, \cdots]=R$. Since $b_j=a_n=c_n \cdots c_{m-1}a_m=c_n \cdots c_{m-1}b_{j+1}$ for appropriate m and n, there is some $n \leqq t \leqq m-1$ such that $c_t \in M$. This is true for all j, hence infinitely many members of $\{c_i\}$ are in M. Q.E.D.

The next two corollaries give rings which satisfy Theorem 2.

COROLLARY 1. *Let R be a domain such that each nonzero principal ideal of R is contained in only finitely many maximal ideals. If R contains a sequence $\{a_i\}$ such that each a_{i+1} properly divides a_i in R, then R is a β-domain.*

PROOF. Let $M_1, \cdots, M_t$ be the complete set of maximal ideals containing the principal ideal (a_1) and suppose that $c_i a_{i+1}=a_i$ for all i, then $\{c_i\} \subset \bigcup_{j=1}^t M_j$. For suppose not, then there is a $c_r \notin \bigcup M_j$. Pick a maximal ideal M of R containing c_r. Since $a_1=c_1c_2 \cdots c_r a_{r+1}$, $a_1 \in M$. Hence M is one of the M_j, a contradiction. Therefore $\{c_i\} \subset \bigcup M_j$ and it follows that infinitely many c_i are in one of the M_j. By Theorem 2, R is a β-domain. Q.E.D.

A nonzero fractional ideal I of the domain R is *divisorial* in case I is the intersection of principal fractional ideals.

COROLLARY 2. *If R is a domain in which each nonzero ideal is divisorial and if R contains a sequence $\{a_i\}_{i=1}^{\infty}$ such that each a_{i+1} properly divides a_i, then R is a β-domain.*

PROOF. Heinzer has proved, [4, Theorem 2.5], that each ideal of R is contained in only a finite number of maximal ideals. Apply Corollary 1. Q.E.D.

3. Intersection theorems.

LEMMA. *Let $\{\alpha_i\}$ be a strictly decreasing sequence of positive real numbers. Then there is a subsequence $\{\beta_j\}$ of $\{\alpha_i\}$ such that $\{\beta_j-\beta_{j+1}\}_{j=1}^{\infty}$ is a strictly decreasing sequence of positive real numbers.*

PROOF. Let $\beta_1=\alpha_1$ and $r_1=\lim_{i=1}^{\infty} \alpha_i$. Let $r_2'=\beta_1-r_1$ and $r_2=r_2'/2+r_1$. Choose $\alpha_{i_2}<r_2$ and define $\beta_2=\alpha_{i_2}$. By induction define $r_j'=\beta_{j-1}-r_1$ and $r_j=r_j'/2+r_1$. Choose $\alpha_{i_j}<r_j$ and define $\beta_j=\alpha_{i_j}$. $\{\beta_j-\beta_{j+1}\}$ is the required sequence. Q.E.D.

For the concepts concerning value groups and valuation rings which are used in the proofs of Theorems 3 and 4, we refer the reader to [1] and [7].

THEOREM 3. *Suppose that R is a β-domain for a rank one valuation v*

which is defined on k, the quotient field of R. Let $\{a_i\}$ be a β-sequence for v in R and let $T=R[a_1/a_2, a_2/a_3, \cdots]$. If w is a valuation on k with valuation ring R_w, then:

(i) *If R_w is noetherian, then $R_w \cap T$ is a β-domain for v;*

(ii) *If R_w is non-noetherian, then $R_w \cap T$ is a β-domain for v or $R_w \cap R_p$ is a β-domain for both v and w with respect to every prime ideal P of R containing $\{a_i\}$;*

(iii) *If a_{i+1} properly divides a_i in R for all i and if $R_w \cap T$ is a β-domain, then $R_w \cap R$ is a β-domain.*

Proof. If R_w contains infinitely many members of the sequence $\{a_i\}$, then by (4) $R_w \cap R$, $R_w \cap T$, and $R_w \cap R_P$ are all β-domains. So we will assume that R_w contains only finitely many elements of $\{a_i\}$. Choose an infinite sequence $\{a_{i'}\}$ of $\{a_i\}$ such that each $a_{i'} \notin R_w$. Again by (4), $\{a_{i'}\}$ is a β-sequence for v in R. Since v is a rank one valuation, its value group is an ordered additive subgroup of the real number system. By the lemma, $\{v(a_{1'})\}$ contains an infinite subsequence $\{v(b_j)\}$ such that $\{v(b_j)-v(b_{j+1})\}$ is a strictly decreasing sequence of positive real numbers. Thus by (5) $\{b_j/b_{j+1}\}$ is a β-sequence for v in T.

Since R_w is a valuation ring and since each $b_j \notin R_w$, we have that b_j^{-1} is a nonunit of R_w. Suppose that R_w is a noetherian valuation ring. Then there is a positive integer $1'$ such that $0 < w(b_{1'}^{-1}) \leqq w(b_j^{-1})$ for all j. Then $b_{1'+1}^{-1} = (b_{1'+1}^{-1}/b_{1'}^{-1})b_{1'}^{-1}$ and $b_{1'+1}^{-1}/b_{1'}^{-1} = b_{1'}/b_{1'+1} \in R_w$. Define $c_1 = b_{1'}/b_{1'+1}$. Let $2'$ be a positive integer $>1'$ such that $0 < w(b_{2'}^{-1}) \leqq w(b_j^{-1})$ for all $j > 1'$. As above let $c_2 = b_{2'}/b_{2'+1} \in R_w$. By induction we construct a sequence $\{c_k\}$ so that $c_k = b_{k'}/b_{k'+1} \in R_w$. Now $v(c_k) - v(c_{k+1}) > 0$ for all k, hence $\{c_k\}$ is a β-sequence for v in $R_w \cap T$. This proves (i).

Now assume that R_w is non-noetherian. Consider the previously constructed sequence $\{b_j\}$. Suppose there exists no strictly decreasing infinite subsequences of $\{w(b_j^{-1})\}$. Then, as in the noetherian case, we can show that $R_w \cap T$ is a β-domain. On the other hand, suppose that $\{b_{j'}\}$ is a subsequence of $\{b_j\}$ such that $w(b_{1'}^{-1}) > w(b_{2'}^{-1}) > \cdots$. Let P be any proper prime ideal of R containing $\{b_j\}$, (certainly any prime ideal P of R containing $\{a_i\}$ will also contain $\{b_j\}$). Clearly $b_{j'}^2+1 \notin P$ for all j', hence each $b_{j'}/(b_{j'}^2+1) \in R_P$. Also $w[b_{j'}/(b_{j'}^2+1)] = -w(b_{j'}) > 0$, which shows that each $b_{j'}/(b_{j'}^2+1) \in R_w \cap R_P$. By straightforward computation we see that $\{b_{j'}/(b_{j'}^2+1)\}$ is a β-sequence for both v and w in $R_w \cap R_P$. This completes the proof of (ii). Part (iii) is clear. Q.E.D.

Our next goal is to prove a version of Theorem 3 for valuations v

of finite rank m, where $m>1$. Recall that if G is the value group of v, then G is order isomorphic to an additive subgroup of $\mathfrak{R}^m$, $\mathfrak{R}$ being the real number system. Order is defined in $\mathfrak{R}^m$ as follows: Let $(\alpha_1, \cdots, \alpha_m)$, $(\beta_1, \cdots, \beta_m) \in \mathfrak{R}^m$, suppose that $\alpha_i = \beta_i$ for $i<p$ and $\alpha_p \neq \beta_p$, then $(\alpha_1, \cdots, \alpha_m) < (\beta_1, \cdots, \beta_m)$ if and only if $\alpha_p < \beta_p$.

In the situation where v is of rank $m>1$, it is necessary to introduce a stronger condition than that of a β-sequence. Let R be a subdomain of R_v. A sequence $\{\alpha_i\}$ of R is called a *β^*-sequence for v in R* in case (1) $\{a_i\}$ is a β-sequence for v in R, and (2) if $v(a_i) = (\alpha_{i1}, \cdots, \alpha_{im})$ then $\alpha_{ij} \geqq 0$ for all i and j. It is clear that if v is a rank one valuation, then the concepts of a β-sequence for v and a β^*-sequence for v are equivalent. However, this is not the case when v is of rank $m>1$. For suppose that v has rank 2 and $\{a_i\}$ is a sequence of elements in R with the property that $v(a_i) = (1, -i)$ for all i. Then $\{a_i\}$ is a β-sequence for v, but not a β^*-sequence for v.[2]

THEOREM 4. *The conclusions of Theorem* 3 *remain valid when v is assumed to be a valuation of finite rank* m>1 *and $\{a_i\}$ is assumed to be a β^*-sequence for v in R.*

PROOF. As in the last proof suppose that $\{a_{i'}\}$ is an infinite subsequence of $\{a_i\}$ such that $a_{i'} \notin R_w$ for each i'. From now on assume that G, the value group of v, is actually a subgroup of $\mathfrak{R}^m$. If r is a positive integer $\leqq m$, then $H_r = \{(\alpha_1, \cdots, \alpha_m) \in G : \alpha_1 = \alpha_2 = \cdots = \alpha_r = 0\}$ is an isolated subgroup of G. All isolated subgroups of G can be obtained in this way. If $H_1, \cdots, H_m$ are the isolated subgroups of G, then $G = H_0 > H_1 > \cdots > H_m = (0)$. It is possible to choose an infinite subsequence $\{d_k\}$ of $\{a_{i'}\}$ such that for some r, $v(d_k) \in H_{r-1}$ and $v(d_k) \notin H_r$ for all k. Then for each k, $v(d_k) = (0, \cdots, 0, \alpha_k^r, \cdots, \alpha_k^m)$ where $\alpha_k^r > 0$. Since $v(d_k) > v(d_{k+1})$, $\alpha_k^r \geqq \alpha_{k+1}^r$. Assume that the sequence $\{\alpha_k^r\}$ has a minimum, then there exists a $t(1)$ such that $\alpha_{t(1)+p}^r = \alpha_{t(1)}^r$ for $p = 1, 2, \cdots$. Consider the infinite sequence $\{\alpha_{t(1)+k}^{r+1}\}_{k=1}^\infty$. If this sequence has a minimum, there is a $t(2) > t(1)$ such that $\alpha_{t(2)}^{r+1} = \alpha_{t(2)+p}^{r+1}$ for $p = 1, 2, \cdots$. Continue this process, since $v(d_k) > v(d_{k+1})$ for all k, we must eventually find a positive integer u such that the sequence $\{\alpha_{t(u)+k}^{r+u}\}_{k=1}^\infty$ does not have a minimum. Pick a strictly decreasing subsequence from $\{\alpha_{t(u)+k}^{r+u}\}$. Since $\{\alpha_i\}$ is a β^*-sequence, each $\alpha_{t(u)+k}^{r+u}$ is a positive real number. Hence it is possible to pick a subsequence $\{\beta_j\}$, from the strictly decreasing sequence already chosen, satisfying the lemma. Let b_j be

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[2] This fact was pointed out to the author by the referee.

the element of $\{a_{i'}\}$ such that $v(b_j)=(0, \cdots, 0, \gamma_j^r, \cdots, \gamma_j^{r+u-1}, \beta_j, \cdots, \gamma_j^m)$. Note that by our construction we have $\gamma_j^r=\gamma_{j+1}^r$, $\gamma_j^{r+1}=\gamma_{j+1}^{r+1}, \cdots, \gamma_j^{r+u-1}=\gamma_{j+1}^{r+u-1}$ for all j. Hence $v(b_j/b_{j+1}) =(0, \cdots, 0, \beta_j-\beta_{j+1}, \cdots, \gamma_j^m-\gamma_{j+1}^m)$ which implies that $v(b_j/b_{j+1}) >v(b_{j+1}/b_{j+2})$, for each j. By (5) each $b_j/b_{j+1}\in T$. Thus $\{b_j/b_{j+1}\}$ is a β-sequence for v in T. To complete the proof proceed exactly as we did in the proof of Theorem 3, (starting with the second paragraph of the proof of Theorem 3).

COROLLARY 3. *Assume the same hypothesis as in Theorem 4. If P is any prime ideal of R containing $\{a_i\}$, then $R_P\cap R_w$ is a β-domain for v.*

PROOF. R_P is a β-domain with β-sequences $\{a_i\}$. If infinitely many a_i are in R_w, then $R_w\cap R_P$ is a β-domain. Assume there are only finitely many $a_i\in R_w$, then we choose an infinite subsequence $\{b_j\}$ of $\{a_i\}$ such that each $b_j\notin R_w$. Let P be a prime ideal of R containing $\{a_i\}$, and hence $\{b_j\}$. Then, as in the proof of Theorem 3, $b_j/(b_j^2+1) \in R_P\cap R_w$ for all j and $\{b_j/(b_j^2+1)\}$ is a β-sequence for v in $R_P\cap R_w$.

We conclude with a remark on regular rings. Auslander and Buchsbaum, [2], defined a noetherian ring R to be regular in case R_P is a regular local ring for each prime ideal P in R. Generalize this by defining a ring R, not necessarily noetherian, to be regular in case R_P is a regular local ring for each prime ideal P in R. Nakano gives, in [6], an example of a regular non-noetherian ring. We were led to the study of β-domains by trying to determine all non-noetherian regular rings. This we have not been able to do. However, β-domains do give some information about non-noetherian regular domains, though negative in character. In fact any β-domain is necessarily nonregular. To see this, suppose that R is a β-domain for v and that $\{a_i\}$ is a β-sequence in R. Let P be a prime ideal of R containing $\{a_i\}$. Then $R\subset R_P\subset R_v$. By (3), R_P is a β-domain and is therefore non-noetherian. Hence R_P is not a regular local ring.

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