ASYMPTOTIC VALUES OF NORMAL LIGHT INTERIOR FUNCTIONS DEFINED IN THE UNIT DISK[1]

J. H. MATHEWS

ABSTRACT. Lehto and Virtanen have extended Lindelöf's theorem for the class of normal meromorphic functions. It is shown that Lindelöf's theorem cannot be extended for the class of bounded normal light interior functions. A generalization of Lindelöf's theorem is given.

1. **Preliminaries.** Let D be the unit disk, C the unit circle. Let f be a light interior function from D into the Riemann sphere W, i.e. f is a continuous open map which does not take any continuum into a single point. It is known that f has a factorization $f=g \circ h$ where h is a homeomorphism of the unit disk onto either the unit disk or the finite complex plane and g is a nonconstant meromorphic function. We will be concerned with the case when the range of h is the unit disk. It is shown that Lindelöf's theorem cannot be extended for the class of bounded normal light interior functions. A generalization of Lindelöf's theorem is given.

A simple continuous curve Γ: $z(t)$ $(0 \leqq t<1)$ contained in D is called a *boundary path* if $|z(t)| \to 1$ as $t \to 1$. The *end* of a boundary path Γ is the set $\bar{\Gamma} \cap C$. A boundary path Γ is an *asymptotic path of f for the value c* provided $f(z(t)) \to c$ as $t \to 1$. The value c is called a *point asymptotic value of f* if there exists an asymptotic path of f for the value c whose end consists of a single point.

The *cluster set*, $C(f, \theta)$, of f at $e^{i\theta}$ is the set of all points $w \in W$ for which there exists a sequence $\{z_n\}$ of points in D with $z_n \to e^{i\theta}$ and $f(z_n) \to w$. By a *Stolz domain* Δ at $e^{i\theta}$ we mean a set of the form

$$\{z \in D: -\pi/2 < \phi_1 < \arg(1 - z/e^{i\theta}) < \phi_2 < \pi/2\},$$

and by a *terminal Stolz domain* at $e^{i\theta}$ we mean a set of the form

$$\Delta \cap \{z: |z - e^{i\theta}| < \epsilon\} \qquad (0 < \epsilon < 1).$$

The function f is said to have the *Fatou value* c at $e^{i\theta}$ if $f(z) \to c$ as $z \to e^{i\theta}$ from within each Stolz domain Δ at $e^{i\theta}$.

Received by the editors July 14, 1969.

AMS Subject Classifications. Primary 3030, 3062.

Key Words and Phrases. Light interior function, normal function, angular limit, asymptotic value.

[1] This paper is part of the author's doctoral thesis directed by Professor Peter Lappan at Michigan State University.

The function f is *normal* if it is uniformly continuous with respect to the non-Euclidean hyperbolic metric ρ in D and the chordal metric in W [1]. Let h be a homeomorphism of D onto D. If h is uniformly continuous with respect to the non-Euclidean hyperbolic metric in both its domain and range then we say that h is HUC. Since the composition of two uniformly continuous functions is uniformly continuous the following theorem is immediate [4].

THEOREM A. *Let h be a homeomorphism of D onto D which is* HUC. *If g is a nonconstant normal meromorphic function, then the light interior function $f = g \circ h$ is normal.*

2. **The main results.** Lindelöf's theorem states that if a bounded holomorphic function possesses the point asymptotic value c at $e^{i\theta}$, then it possesses the Fatou value c at $e^{i\theta}$. The following result shows that a bounded normal light interior function can possess point asymptotic values at almost every point of C and possess no Fatou values.

THEOREM 1. *There exists a bounded normal light interior function which possesses point asymptotic values at almost every point of C, but which possesses no Fatou values.*

The following lemma is used in the proof of the theorem.

LEMMA 1. *There exists a homeomorphism h of $\overline{D}$ onto $\overline{D}$ with the following properties:*

(a) *the radius τ_0 at $z=1$ is mapped onto an arc Γ_0, where Γ_0 is a Jordan arc lying in $D \cup \{1\}$ internally tangent to C at $z=1$, and having the origin as the other end point,*

(b) *if Γ_θ denotes the image of Γ_0 under a rotation through an angle θ about the origin, then the radius τ_θ at $e^{i\theta}$ is mapped onto Γ_θ,*

(c) *the restriction of h to C is the identity function, and*

(d) *h is* HUC *in D.*

PROOF. Let $\{R_n\}$ be a strictly increasing sequence of nonnegative real numbers with $R_0=0$ and $\rho(R_n, R_{n+1}) = 1/(1-R_n^2)$. Let $\Phi(r)$ be a mapping of the interval $[0, 1)$ onto itself defined by $\Phi(r) = (rR_2)/R_3$ for $0 \leqq r < R_3$ and satisfying the equation $\rho(R_{n-1}, \Phi(r))/\rho(R_{n-1}, R_n) = \rho(R_n, r)/\rho(R_n, R_{n+1})$ for $R_n \leqq r < R_{n+1}$ $(n=3, 4, \cdots)$. A straightforward calculation shows that if

$$r \leqq r' \quad \text{then} \quad \rho(\Phi(r), \Phi(r')) \leqq \rho(r, r').$$

Let $C^* = \{z: \operatorname{Im} z > 0, |z - \frac{1}{2}| = \frac{1}{2}\}$ and $C_n = \{z: |z| = R_n\}$. Let $C^* \cap C_n = w_n$ and let $\alpha_n = \arg(w_n)$ $(n=1, 2, \cdots)$, and let $\alpha_0 = 0$. Then

$\{\alpha_n\}$ is a sequence of positive real numbers with $\alpha_n \to 0$. Define a function $\Psi(r)$ on $[0, 1)$ by $\Psi(r) = \pi/2 + [\alpha_1 - \pi/2]\rho(0, r)/\rho(0, R_3)$ for $0 \leqq r < R_3$ and satisfying the equation

$$\Psi(r) = \alpha_{n-2} + [\alpha_{n-1} - \alpha_{n-2}]\rho(R_n, r)/\rho(R_n, R_{n+1})$$

for $R_n \leqq r < R_{n+1}$ $(n = 3, 4, \cdots)$.

Let the mapping h in $\overline{D}$ be defined by

$$h(z) = h(re^{i\theta}) = \Phi(r) \exp(i\theta + i\Psi(r))$$

for $0 \leqq r < 1$ and set $h(e^{i\theta}) = e^{i\theta}$. It is easy to verify that h is a homeomorphism of $\overline{D}$ onto $\overline{D}$ and that the radius τ_0 is mapped onto a Jordan arc Γ_0 which is internally tangent to C at the point $z = 1$.

Set $A_n = \{z: R_n \leqq |z| < R_{n+1}\}$. Let $n \geqq 2$ be fixed but arbitrary and let $z, z' \in A_n$ with $\rho(z, z') < 1$. It suffices to show that we can find a constant K, independent of n, for which $\rho(h(z), h(z')) \leqq K\rho(z, z')$. The case when $z \in A_n$ and $z' \notin A_n$ can then easily be disposed of by introducing an appropriate number of intermediate points on the geodesic between z and z' and lying on the circles C_j, and adding the inequalities thus obtained. We may assume that $z = re^{i\alpha}$ and $z' = r'e^{i\beta}$ with $r \leqq r'$. Then we have the following inequality

$$\begin{aligned}\rho(h(z), h(z')) \leqq\ & \rho(\Phi(r) \exp(i\alpha + i\Psi(r)), \Phi(r) \exp(i\beta + i\Psi(r))) \\ & + \rho(\Phi(r) \exp(i\beta + i\Psi(r)), \Phi(r) \exp(i\beta + i\Psi(r'))) \\ & + \rho(\Phi(r) \exp(i\beta + i\Psi(r')), \Phi(r') \exp(i\beta + i\Psi(r'))).\end{aligned}$$

From the fact that $\Phi(r) \leqq r$ we obtain

$$\begin{aligned}&\rho(\Phi(r) \exp(i\alpha + i\Psi(r)), \Phi(r)\exp(i\beta + i\Psi(r))) \\ &\qquad = \rho(\Phi(r)e^{i\alpha}, \Phi(r)e^{i\beta}) \leqq \rho(re^{i\alpha}, re^{i\beta}) \leqq \rho(z, z').\end{aligned}$$

From the facts that $\Phi(r) \leqq R_n$, $\rho(R_n, R_{n+1}) = 1/(1 - R_n)$ and $|\alpha_{n-1} - \alpha_{n-2}| < \pi/2$ we obtain

$$\begin{aligned}&\rho(\Phi(r) \exp(i\beta + i\Psi(r)), \Phi(r) \exp(i\beta + i\Psi(r'))) \\ &\qquad \leqq \left| \int_{\Psi(r)}^{\Psi(r')} \frac{\Phi(r)d\theta}{1 - [\Phi(r)]^2} \right| \\ &\qquad \leqq \pi[\rho(R_n, r') - \rho(R_n, r)]/2[(1 - R_n^2)\rho(R_n, R_{n+1})] \\ &\qquad \leqq \pi\rho(r, r')/2 \leqq \pi\rho(z, z')/2.\end{aligned}$$

From the fact that $\rho(\Phi(r), \Phi(r')) \leqq \rho(r, r')$ we obtain

$$\begin{aligned}&\rho(\Phi(r) \exp (i\beta + i\Psi(r')), \Phi(r') \exp(i\beta + i\Psi(r'))) \\ &\qquad = \rho(\Phi(r), \Phi(r')) \leqq \rho(r, r') \leqq \rho(z, z').\end{aligned}$$

Combining the above estimates we choose $K=2+\pi/2$ and the proof of the lemma is complete.

PROOF OF THEOREM 1. Let h be the homeomorphism and Γ_0 be the Jordan arc of Lemma 1. By a theorem of Lohwater and Piranian [3, Theorem 9, p. 15], there exists a bounded holomorphic function g which does not approach a limit as z approaches $e^{i\theta}$ along Γ_θ $(0\leqq\theta<2\pi)$. Hence the bounded light interior function $f=g\circ h$ possesses no radial limits. By Theorem A, f is normal.

Since g is bounded, g possesses radial limits at almost every point of C. Let τ_θ be the radius terminating at $e^{i\theta}$. It follows easily that f has point asymptotic values at almost every point of C along the paths $h^{-1}(\tau_\theta)$; and the theorem is proved.

Let h be a homeomorphism of D onto D. If for every $e^{i\theta}\in C$ and every Stolz domain Δ at $e^{i\theta}$ the image of some terminal Stolz domain of Δ is contained in a Stolz domain, then we say that h *weakly preserves Stolz domains.*

THEOREM 2. *Let f be a light interior function with factorization $f=g\circ h$ where both h and h^{-1} weakly preserve Stolz domains and g is a nonconstant normal meromorphic function. If f has the point asymptotic value c at $e^{i\theta}$, then f has the Fatou value c at $e^{i\theta}$.*

Before we prove the theorem we establish the following lemma.

LEMMA 2. *If h is a homeomorphism of D onto D for which both h and h^{-1} weakly preserve Stolz domains, then h can be extended to a homeomorphism of $\overline{D}$ onto $\overline{D}$.*

PROOF. Suppose h cannot be extended to be continuous in $\overline{D}$. Then there exists a point $e^{i\theta}$ for which $C(h,\theta)=[\phi_1,\phi_2]$, with $0<\phi_2-\phi_1\leqq 2\pi$. There exist two radii τ_1 and τ_2 terminating at $e^{i\alpha}$ and $e^{i\beta}$, respectively, with $\phi_1<\alpha<\beta<\phi_2$ for which $h^{-1}(re^{i\alpha})\to e^{i\theta}$ and $h^{-1}(re^{i\beta})\to e^{i\theta}$. Since h weakly preserves Stolz domains, h has exactly one nontangential asymptotic value at $e^{i\theta}$, and hence one of $h^{-1}(\tau_1)$ and $h^{-1}(\tau_2)$ must be a tangential asymptotic path. But then h^{-1} does not weakly preserve Stolz domains which contradicts our hypothesis. Therefore h can be extended to be continuous in $\overline{D}$ and similarly h^{-1} can be extended to be continuous in $\overline{D}$. It follows easily that h can be extended to a homeomorphism of $\overline{D}$ onto $\overline{D}$ and the lemma is proved.

PROOF OF THEOREM 2. From Lemma 2, h can be extended to a homeomorphism of $\overline{D}$ onto $\overline{D}$. Let f have the point asymptotic value c along an asymptotic path Γ terminating at $e^{i\theta}$. Then $h(\Gamma)$ is an asymptotic path terminating at $h(e^{i\theta})$ along which g has the asymptotic value c. By a theorem of Lehto and Virtanen [2, Theorem 2, p. 53], g has the Fatou value c at $h(e^{i\theta})$. Since h weakly preserves Stolz domains it follows easily that f has the Fatou value c at $e^{i\theta}$ and the proof is complete.

Theorem 1 shows that the hypothesis that h is merely a homeomorphism of $\bar{D}$ onto $\bar{D}$ is not sufficient to imply the conclusion of Theorem 2. However, by a theorem of Mori [5, Theorem 6, p. 69], if h is a K-quasiconformal homeomorphism of D onto D, then both h and h^{-1} weakly preserve Stolz domains. Thus we obtain the following result which was first proved by Väisälä [6, Theorem 8, p. 22].

COROLLARY. *Let f be a light interior function in D with factorization $f=g\circ h$ with h a K-quasiconformal homeomorphism of D onto D and g a nonconstant normal meromorphic function. If f has the point asymptotic value c at $e^{i\theta}$, then f has the Fatou value c at $e^{i\theta}$.*

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MICHIGAN STATE UNIVERSITY AND
CALIFORNIA STATE COLLEGE AT FULLERTON