A PROPERTY OF THE RAYLEIGH FUNCTION

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1. **Introduction.** Let $J_\nu(z)$ denote the Bessel function of the first kind. Kishore [3], [4], has defined the Rayleigh function of order $2n$ by means of

$$\sigma_{2n}(\nu) = \sum_{m=1}^{\infty} (j_{\nu,m})^{-2n} \qquad (n = 1, 2, \cdots)$$

where the $j_{\nu,m}$ are the zeros of $z^{-\nu}J_\nu(z)$, $|R(j_{\nu,m})| \leqq |R(j_{\nu,m+1})|$. The Rayleigh function is a rational function of ν and the following recurrence formula has been developed for it [3]:

$$(1.1) \qquad (\nu + n)\sigma_{2n}(\nu) = \sum_{k=1}^{n-1} \sigma_{2k}(\nu)\sigma_{2n-2k}(\nu).$$

It would be of some interest to know exactly which primes divide the numerator or denominator of $\sigma_{2n}(\nu)$, if ν is rational and $\sigma_{2n}(\nu)$ is reduced to its lowest terms. Restricting ourselves to the prime 2, and using induction on (1.1), we shall find in this paper the exact power of 2 dividing $\sigma_{2n}(\nu)$ when ν is a rational number a/b, a odd and b even. This will extend the results of [2] where the same problem was solved for ν one half an odd integer. In this paper we shall also prove some congruences (mod 2), (mod 4), and (mod 8). We note that this type of problem has been considered for other sequences of rational numbers. For the well-known Bernoulli numbers B_{2n}, for example, it has been proved [1] that

$$\begin{aligned} 2B_{2n} &\equiv 1 \pmod{4} (n > 1), \\ &\equiv 1 + 4n \pmod{8} (n > 1), \\ &\equiv 1 + 4n \pmod{16} (n > 2). \end{aligned}$$

Throughout this paper we shall assume a is an odd integer, b is an even integer, $b = (2k+1)2^t$, $t > 0$, and a/b has been reduced to its lowest terms.

2. **The power of 2 dividing the Rayleigh function.**

DEFINITION 2.1. Define $\theta_{2n}(\nu)$ as the exponent of the highest power of 2 dividing the denominator of $\sigma_{2n}(\nu)$. If $\theta_{2n}(\nu)$ is negative, it is under-

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stood that $-\theta_{2n}(\nu)$ is the exponent of the highest power of 2 dividing the numerator of $\sigma_{2n}(\nu)$.

Since we shall be proving our main results by using induction on (1.1), it is useful to list here the first three values of $\sigma_{2n}(a/b)$.

$$\sigma_2(a/b) = b/4(a+b),$$
$$\sigma_4(a/b) = b^3/2^4(a+b)^2(a+2b),$$
$$\sigma_6(a/b) = b^5/2^5(a+b)^3(a+2b)(a+3b).$$

THEOREM 2.1. *Suppose a is odd, $b=(2k+1)2^t$, $t>0$, and*

$$2n = 2^{s_1} + \cdots + 2^{s_m}, \qquad s_1 > \cdots > s_m > 0.$$

Then

$$\theta_{2n}(a/b) = 2n + 1 + (1-2n)t - m.$$

PROOF. The theorem is true for $n=1, 2, 3$. Assume it is true for $1, \cdots, n-1$. Let $x=2n+1+(1-2n)t-m$. From (1.1) we have

$$2^x(a+bn)\sigma_{2n}(a/b) = 2^x b \sum_{k=1}^{[n/2]} \alpha_k \sigma_{2k}(a/b)\sigma_{2n-2k}(a/b) \tag{2.1}$$

where $\alpha_k=1$ if $n=2k$, $\alpha_k=2$ if $n\neq 2k$. Suppose $m>1$. For a fixed k write $2k=2^{r_1}+\cdots+2^{r_u}$ and $2n-2k=2^{q_1}+\cdots+2^{q_w}$. By our induction hypothesis, the exponent of the highest power of 2 dividing the numerator of $2^x b\alpha_k\sigma_{2k}(a/b)\sigma_{2n-2k}(a/b)$ is $u+w-m$ (or $m-1$ if $n=2k$). Hence all terms on the right side of (2.1) are congruent to 0 (mod 2) except those for which

$$2k = 2^{r_1} + \cdots + 2^{r_u}, \qquad 2n-2k = 2^{q_1} + \cdots + 2^{q_w}, \qquad u+w=m. \tag{2.2}$$

There are $C_{m,1}$ such terms with $u=1$, $w=m-1$; $C_{m,2}$ terms with $u=2$, $w=m-2$; etc. It is easily seen there is a total of $2^{m-1}-1$ terms satisfying conditions (2.2). Hence there are an odd number of terms on the right side of (2.1) congruent to 1 (mod 2). Thus we have

$$2^x\sigma_{2n}(a/b) \equiv 1 \pmod 2.$$

If $m=1$, we need only consider the term on the right of (2.1) such that $2n-2k=2k=n$ and the theorem follows.

3. **Congruences** (mod 4) **and** (mod 8). If $2n=2^u$, $u>1$, it is easy to raise the modulus in Theorem 2.1 to 4 and to 8. Again we assume that $b=(2k+1)2^t$, $t>0$.

THEOREM 3.1. *If* $2n=2^u$, *we have*

$$2^{\theta_{2n}(a/b)}\sigma_{2n}(a/b) \equiv (2k+1)/a \pmod 4 \qquad (u>1),$$
$$\equiv 5(2k+1)/a \pmod 8 \qquad (u>2).$$

PROOF. To prove the congruence (mod 8), we use Theorem 2.1 and formula (2.1). Note that $n=2^{u-1}$, $n/2=2^{u-2}$, $3n/2=2^{u-1}+2^{u-2}$. We have

$$\begin{aligned} 2^{\theta_{2n}(a/b)}\sigma_{2n}(a/b) &\equiv \left(\frac{2k+1}{a}\right)(2^{\theta_n(a/b)}\sigma_n(a/b))^2 \\ &\quad + 4(2^{\theta_{3n/2}(a/b)}\sigma_{3n/2}(a/b)\cdot 2^{\theta_{n/2}(a/b)}\sigma_{n/2}(a/b)) \\ &\equiv 5(2k+1)/a \pmod 8. \end{aligned}$$

The next two theorems follow in much the same way from Theorem 2.1 and formula (2.1).

THEOREM 3.2. *If* $2n=2^u+2^v$, *then*

$$2^{\theta_{2n}(a/b)}\sigma_{2n}(a/b) \equiv (2k+1)/a \pmod 4 \qquad (u=v+1,\ v\geqq 1),$$
$$\equiv 3(2k+1)/a \pmod 4 \qquad (u-v>1,\ v\geqq 1).$$

THEOREM 3.3. *If* $2n=2^u+2^v+2^w$, *then*

$$\begin{aligned} &2^{\theta_{2n}(a/b)}\sigma_{2n}(a/b) \\ &\equiv (2k+1)/a \pmod 4 \qquad (u=w+2,\ v=w+1,\ w\geqq 1), \\ &\equiv (2k+1)/a \pmod 4 \qquad (u-v>1,\ v-w>1,\ w\geqq 1), \\ &\equiv 3(2k+1)/a \pmod 4 \qquad (u>3,\ v=2,\ w=1), \\ &\equiv 3(2k+1)/a \pmod 4 \qquad (u-v=1,\ v-w>1,\ w\geqq 1). \end{aligned}$$

In order to find more general congruences (mod 4) we need the following definition.

DEFINITION 3.1. Let $2n=2^{s_1}+\cdots+2^{s_m}$, $s_1>\cdots>s_m>0$. Define $\psi(2n)$ as the number of positive integers $2k$, $2k\leqq n$, such that

$$2k = 2^{r_1}+\cdots+2^{r_u}, \qquad r_1>\cdots>r_u>0,$$
$$2n-2k = 2^{q_1}+\cdots+2^{q_w}, \qquad q_1>\cdots>q_w>0,$$
$$u+w=m+1.$$

The following lemma is important in the proofs of Theorems 3.4–3.7.

LEMMA 3.1. *Let* $2n=2^{s_1}+\cdots+2^{s_m}$, $s_1>\cdots>s_m>0$. *Suppose in this expansion there are* h_1 *"single" terms* 2^{s_i} *such that*

$$s_{i-1} - 1 > s_i > s_{i+1} + 1,$$

h_2 "doubles" 2^{s_i}, $2^{s_{i+1}}$ such that

$$s_{i-1} - 1 > s_i = s_{i+1} + 1 > s_{i+2} + 2,$$

h_3 "triples," etc. Then

$$\psi(2n) = 2^{m-2} \sum_{i=1}^{m} h_i \qquad \text{if } s_m > 1,$$

$$= 2^{m-2}\left(\sum_{i=1}^{m} h_i - 1\right) \qquad \text{if } s_m = 1.$$

We shall continue to use the terms "single," "double," and "triple" in the sense of Lemma 3.1.

THEOREM 3.4. *Let $2n=2^{s_1}+ \cdots +2^{s_m}$, $s_1> \cdots >s_m \geqq 1$. If each term of this expansion is "single" in the sense of Lemma 3.1 ($s_i-s_{i+1}>1$ for $i=1, \cdots, m-1$), then*

$$2^{\theta_{2n}(a/b)}\sigma_{2n}(a/b) \equiv (-1)^{m+1}(2k + 1)/a \pmod 4.$$

PROOF. The proof is by induction on m. The theorem is true for $m=1$, 2, 3. Assume it is true for 1, 2, $\cdots$, $m-1$, and let $2n=2^{s_1} + \cdots +2^{s_m}$. By Lemma 3.1, we know $\psi(2n)\equiv 0 \pmod 2$. Hence in formula (2.1) we need only consider terms on the right side satisfying (2.2). There are $2^{m-1}-1$ such terms. Let $g(2n)$ be the number of these terms congruent to $-1 \pmod 4$. Then, if $s_m>1$,

$$2^{\theta_{2n}(a/b)}\sigma_{2n}(a/b) \equiv \frac{(2k + 1)}{a} \left[2^{m-1} - 1 - 2g(2n)\right] \pmod 4. \tag{3.1}$$

If m is even, then, by our induction hypothesis, $g(2n)=0$. If m is odd, then $g(2n)=2^{m-1}-1$, and the theorem follows for $s_m>1$. If $s_m=1$ the proof is the same, except that in formula (3.1) $(2k+1)/a$ must be replaced by $(2k+1)/(a+2^t)$, and it must be remembered that $2^{\theta_2(a/b)}\sigma_2(a/b)\equiv(2k+1)/(a+2^t) \pmod 4$.

The proofs of Theorems 3.5–3.7 are similar.

THEOREM 3.5. *Let $2n=2^{s_1}+ \cdots +2^{s_m}$, $s_1> \cdots >s_m\geqq 1$. Suppose in this expansion there are $m-2$ "singles" and one "double." Then*

$$2^{\theta_{2n}(a/b)}\sigma_{2n}(a/b) \equiv (-1)^{m}(2k + 1)/a \pmod 4.$$

THEOREM 3.6. *Let $2n=2^{s_1}+ \cdots +2^{s_m}$, $s_1> \cdots >s_m\geqq 1$. If in this expansion there are one "triple" and $m-3$ "singles," then*

$$2^{\theta_{2n}(a/b)}\sigma_{2n}(a/b) \equiv (-1)^{m+1}(2k + 1)/a \pmod 4.$$

THEOREM 3.7. *Let* $2n = 2^{s_1} + \cdots + 2^{s_m}$, $s_1 > \cdots > s_m \geqq 1$. *If in this expansion there are* h *"doubles" (*h *pairs satisfying the definition in Lemma* 3.1), *and* $m - 2h$ *"singles," then*

$$2^{\theta_{2n}(a/b)}\sigma_{2n}(a/b) \equiv (-1)^{h+m+1}(2k+1)/a \pmod 4.$$

4. A conjecture.

The results of Theorems 3.4–3.7 lead us to the following conjecture.

CONJECTURE. Let $2n = 2^{s_1} + \cdots + 2^{s_m}$, $s_1 > \cdots > s_m \geqq 1$. If in this expansion there are h_1 "singles," h_2 "doubles," h_3 "triples," etc., then letting $\sum_1^m h_i = h$, we have

$$2^{\theta_{2n}(a/b)}\sigma_{2n}(a/b) \equiv (-1)^{h+1}(2k+1)/a \pmod 4.$$

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