EXAMPLES OF NONNORMAL SEMINORMAL OPERATORS WHOSE SPECTRA ARE NOT SPECTRAL SETS

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ABSTRACT. An example is given of a nonnormal seminormal operator on a Hilbert space whose spectrum is thin (in the sense of von Neumann) and is therefore not a spectral set. It is shown that every nonnormal subnormal operator is the limit of a sequence of hyponormal and nonsubnormal operators.

1. **Introduction.** An operator T on a Hilbert space $\mathcal{H}$ is said to be semi-normal in case its selfadjoint selfcommutator $T^*T-TT^*=D$ is positive semidefinite ($D\geqq 0$) or negative semidefinite ($D\leqq 0$). In the case $D\geqq 0$ the operator T is called hyponormal. An interesting subclass of the hyponormal operators is the class of subnormals: an operator T on $\mathcal{H}$ is said to be subnormal in case T is the restriction of a normal operator A acting on a superspace $\mathfrak{N}\supset\mathcal{H}$.

It is known that the spectrum of a subnormal operator is a spectral set (see, e.g., Lebow [**5**]). Moreover, Bishop [**2**] has characterized the subnormal operators as precisely the closure, in the strong operator topology, of the normal operators (see also Stampfli [**10**]).

In this note an example is given of a seminormal operator whose spectrum is not a spectral set (§3). This example motivates a construction which shows that every nonnormal subnormal operator is a strong limit of a sequence of hyponormal and nonsubnormal operators (§4).

2. **Preliminaries.** If X is a compact set in the plane, then $C(X)$ will denote the algebra of all complex continuous functions on X with norm defined by $\|f\|_X=\sup\{|f(z)|:z\in X\}$ for f in $C(X)$. The symbols $R(X)$ and $P(X)$ will be used to denote, respectively, the set of restrictions to X of the rational functions without poles in X and the polynomial functions. The closures of $R(X)$ and $P(X)$ in $C(X)$ will be denoted by $\mathrm{Cl}(R(X))$ and $\mathrm{Cl}(P(X))$.

The spectrum of an operator T is denoted by $\sigma(T)$. If X is a set in the plane, $\mathrm{pr}_x(X)$ and $\mathrm{pr}_y(X)$ will be used to denote the projections of X on the x and y-axes. The notations meas_1 and meas_2 will be employed for linear and planar Lebesgue measure, respectively.

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A compact set X in the plane is called a spectral set for an operator T in case $\|f(T)\| \leqq \|f\|_X$ for all $f \in R(X)$.

The following theorem of von Neumann [**7**, p. 279, Satz 6.5] is also proved in Lebow [**5**]:

THEOREM vN. *If* $\mathrm{Cl}(R(X)) = C(X)$ *and X is a spectral set for the operator T, then T is normal.*

The following approximation theorem is due to Lavrentieff [**4**] (cf. Wermer [**11**, p. 74, Theorem 7.3] and Rudin [**9**, p. 386]):

THEOREM L. *If X is a compact set in the plane having no interior and such that the complement of X is connected, then* $\mathrm{Cl}(P(X)) = C(X)$.

The next two theorems appear in Putnam [**8**, p. 46 and p. 54]:

THEOREM P1. *If $T = H + iJ$ is the Cartesian decomposition of the seminormal operator T, then* $\mathrm{pr}_x(\sigma(T)) = \sigma(H)$ *and* $\mathrm{pr}_y(\sigma(T)) = \sigma(J)$.

THEOREM P2. *If $T = H + iJ$ is seminormal and $\sigma(H)$ contains no interval, then* $\|T^*T - TT^*\| \leqq (1/\pi)$ $\mathrm{meas}_2\, \sigma(T)$.

An operator T is called normaloid if $\sup\{|\lambda| : \lambda \in \sigma(T)\} = \|T\|$.

Berberian [**1**] has given the following characterization of operators whose spectrum is a spectral set:

THEOREM B. *The spectrum $\sigma(T)$ is a spectral set for the operator T if and only if $f(T)$ is normaloid for every $f \in R(\sigma(T))$.*

Finally, if H is selfadjoint, with spectral resolution $H = \int \lambda dE_\lambda$, then H is said to be absolutely continuous in case $\|E_\lambda x\|^2$ is an absolutely continuous function of λ, for each $x \in H$.

3. **The example.** Kato [**3**] has given examples of seminormal operators whose real parts have spectra that are Cantor sets of positive measure. The following example is of the type studied by Kato.

Let K be a bounded real Cantor set of positive measure; such a set K is perfect in measure, that is, every neighborhood of a point in K intersects K in a set of positive measure. For $f \in L^2(K)$ (with respect to Lebesgue measure), define the operator

$$Tf(s) = sf(s) + (1/\pi) \int_K f(t)(s - t)^{-1} dt \qquad (s \in K).$$

If the singular integral is interpreted as a Cauchy principal value, then the operator T is bounded (see, e.g., Muskhelišvili [**6**]). Moreover, $(T^*T - TT^*)f = (2/\pi)(f, 1_K)1_K$, where 1_K denotes the characteristic function of K, so that T is hyponormal and nonnormal. Clearly $\|T^*T-TT^*\|=(2/\pi)$ $\text{meas}_1 K$.

The selfadjoint operators H and J, defined by $Hf(s)=sf(s)$ and $Jf(s)=(1/\pi i)\int_K f(t)(s-t)^{-1}dt$ for $s\in K$ and $f\in L^2(K)$, are the real and imaginary parts of T, respectively. It is known that $\sigma(H)=K$, and that $\sigma(J)=[-1, 1]$ (see, e.g., Putnam [**8**, p. 140]); it follows from Theorem P1 that $\sigma(T)\subset K\times[-1, 1]$, therefore $\text{meas}_2\, \sigma(T) \leqq 2\, \text{meas}_1 K$. On the other hand, since $\|T^*T-TT^*\|=(2/\pi)\, \text{meas}_1 K$, Theorem P2 yields $2\, \text{meas}_1 K\leqq \text{meas}_2\, \sigma(T)$. Thus $\text{meas}_2\, \sigma(T) = 2\, \text{meas}_1 K = \text{meas}_2(K\times[-1, 1])$. Since K is perfect in measure one easily concludes that $\sigma(T)=K\times[-1, 1]$.

Then $\sigma(T)$ is a compact set in the plane without interior and with connected complement, hence by Theorem L $\text{Cl}(P(\sigma(T))) = \text{Cl}(R(\sigma(T)))=C(\sigma(T))$. It follows that $\sigma(T)$ is not a spectral set for T (if it were, Theorem vN would imply T normal). The operator T cannot be subnormal since the spectrum of a subnormal operator is a spectral set [**5**, p. 82].

From Theorem B and a lemma of Lebow [**5**, p. 66] an additional interesting property of our example T can be concluded: for some polynomial p the operator $p(T)$ is nonnormaloid.

4. **The construction.** Let $T=H+iJ$ be hyponormal and nonnormal, so that $(Dx_0, x_0)>0$ for some x_0 in $\mathfrak{H}$. Assume H is absolutely continuous with spectral measure $E(\beta)$ for β a Borel set.

Choose an increasing sequence $\{\beta_n\}$ of perfect nowhere dense sets in $\sigma(H)$ such that $\text{meas}_1(\beta_n)\to\text{meas}_1(\sigma(H))$. Then since H is absolutely continuous, $E(\beta_n)x\to x$ for all x in $\mathfrak{H}$. From the identity $D=2i[HJ-JH]$ it easily follows that the operators $T_\beta=E(\beta)TE(\beta)$ are hyponormal on $\mathfrak{H}$. Moreover, $\lim T_{\beta_n}x=Tx$ for all x. Further for n sufficiently large $(E(\beta_n)DE(\beta_n)x_0, x_0)$ must be positive. Since $T^*_{\beta_n}T_{\beta_n} - T_{\beta_n}T^*_{\beta_n}=E(\beta_n)DE(\beta_n)$, then for large n, T_{β_n} will be hyponormal and nonnormal. It follows from Theorem P1 that $\sigma(T_{\beta_n})\subseteq \beta_n\times[-1,1]\cup\{0\}$.

As in §3 it is possible to conclude, when n is large, that $\sigma(T_{\beta_n})$ is not a spectral set for T_{β_n} and hence T_{β_n} is hyponormal and nonsubnormal.

The requirement that H be absolutely continuous can be removed in virtue of the following result of Putnam [**8**, p. 42]:

THEOREM P3. *Let $T=H+iJ$ be hyponormal. Let $\mathfrak{M}=\mathfrak{M}_T$ denote the smallest subspace of H which is invariant under T and T^* and contains the range of $D=T^*T-TT^*$. Then $\mathfrak{M}$ reduces T and if $T'=H'+iJ'$ is* *the restriction of T to $\mathfrak{M}$ then H' is absolutely continuous. Moreover, T on the orthogonal complement of $\mathfrak{M}$ is a normal operator.*

From the above construction and Theorem P3 there follows:

THEOREM C. *Let T be subnormal and nonnormal. Then T is the strong limit of a sequence of hyponormal and nonsubnormal operators.*

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