OPERATOR-VALUED FEYNMAN INTEGRALS OF CERTAIN FINITE-DIMENSIONAL FUNCTIONALS

G. W. JOHNSON AND D. L. SKOUG

ABSTRACT. Let $C_0[a, b]$ denote the space of continuous functions x on $[a, b]$ such that $x(a)=0$. Let $F(x)=f_1(x(t_1)) \cdots f_n(x(t_n))$ where $a=t_0<t_1< \cdots <t_n=b$. Recently, Cameron and Storvick defined an operator-valued "Feynman Integral." In their setting, we give a strong existence theorem as well as an explicit formula for the "Feynman Integral" of functionals F as above under weak restrictions on the f_j's. We also give necessary and sufficient conditions for the operator to be invertible and an explicit formula for the inverse.

1. Introduction and notation. Let $\mathfrak{L}(L_2)$ be the space of bounded linear operators on $L_2=L_2(-\infty, \infty)$. Let $B[a, b]$ be the space of functions on $[a, b]$ which are continuous except for a finite number of finite jump discontinuities.

We need the following definitions of Cameron and Storvick [2]. (The definitions are not intended to imply the existence of the operators involved. In fact the main theorems of [2] give various conditions on F insuring the existence of these operators.) Let $\psi \in L_2$, $\xi \in (-\infty, \infty)$, and F a functional on $B[a, b]$. For $\lambda>0$, $I_\lambda(F)$ is the operator on L_2 defined by the Wiener integral

$$(1) \qquad (I_\lambda(F)\psi)(\xi) = \int_{C_0[a,b]} F(\lambda^{-1/2}x + \xi)\psi(\lambda^{-1/2}x(b) + \xi)dx.$$

For $\operatorname{Re}\lambda>0$, $I_\lambda^{\mathrm{an}}(F)$ is defined to be the operator-valued function of λ which agrees with $I_\lambda(F)$ for $\lambda>0$ and is analytic throughout $\operatorname{Re}\lambda>0$. For $\operatorname{Re}\lambda>0$ and any partition σ: $a=s_0<s_1< \cdots <s_n=b$ the operator $I_\lambda^\sigma(F)$ is defined by the formula

$$(2) \qquad \begin{aligned}(I_\lambda^\sigma(F)\psi)(\xi) = \lambda^{n/2}[(2\pi)^n(s_1-a)\cdots(s_n-s_{n-1})]^{-1/2}\int_{-\infty}^{\infty}\cdot\overset{(n)}{\underset{2}{\cdots}}\\ \cdot\int_{-\infty}^{\infty}\psi(v_n)\cdot f_\sigma(\xi, v_1, \cdots, v_n)\exp\left(-\sum_{j=1}^{n}\frac{\lambda(v_j-v_{j-1})}{2(s_j-s_{j-1})}\right)dv_1\cdots dv_n\end{aligned}$$

Received by the editors June 24, 1969.

AMS Subject Classifications. Primary 2825, 4690.

Key Words and Phrases. Feynman integral, Wiener integral, convolution operator, Fourier transform, operator valued Feynman integral.

where $v_0=\xi$, $f_\sigma(v_0, v_1, \cdots, v_n)=F[z(\sigma, v_0, v_1, \cdots, v_n, \cdot)]$ and

$$
\begin{aligned}
z(\sigma, v_0, v_1, \cdots, v_n, s) &= v_j \qquad \text{if } s_j \leqq s < s_{j+1}, j = 0, 1, \cdots, n-1 \\
&= v_n \qquad \text{if } s = b.
\end{aligned}
$$

(If n is odd we always choose $\lambda^{1/2}$ with nonnegative real part.) For $\operatorname{Re}\lambda>0$, the operator $I_\lambda^{\mathrm{seq}}(F)$ is defined by

$$I_\lambda^{\mathrm{seq}}(F) = w \lim_{\operatorname{norm}\sigma\to 0} I_\lambda^\sigma(F)$$

where w lim means the limit with respect to the weak operator topology on $\mathcal{L}(L_2)$. In case both $I_\lambda^{\mathrm{an}}(F)$ and $I_\lambda^{\mathrm{seq}}(F)$ exist and agree we will denote their common value by $I_\lambda(F)$. Finally for $\lambda=-iq$, $q\in(-\infty, \infty)$, $q\neq 0$, the operators $J_q^{\mathrm{an}}(F)$ and $J_q^{\mathrm{seq}}(F)$ are defined by

$$J_q^{\mathrm{an}}(F) = w \lim_{p\to 0^+} I_{p-iq}^{\mathrm{an}}(F) \quad \text{and} \quad J_q^{\mathrm{seq}}(F) = w \lim_{p\to 0^+} I_{p-iq}^{\mathrm{seq}}(F).$$

Again if both exist and agree we denote their common value by $J_q(F)$.

Throughout the rest of this paper we assume that F has the special form given in the abstract, where each f_j is measurable and satisfies $\|f_j\|_\infty<\infty$. This restriction on the f_j's is much weaker than in [**1**, pp. 333–348] and [**5**, pp. 177–185] where Cameron's earlier definition of the Feynman and related integrals was employed to study functionals of the same type.

The main theorem below does more than establish the existence of $J_q(F)$ and give a formula for it. First, it shows that $J_q(F)$ is the strong operator limit of $I_\lambda(F)$ rather than just the weak operator limit; secondly, the approach of λ to $-iq$ is restricted only to the open right half plane and not to the line $p-iq$. We mention also that the existence of $J_q(F)$ is established for every $q\neq 0$. In [**2**], where more complicated functionals F are dealt with, the theorems give existence of $J_q(F)$ for almost every q, but for specific q, one cannot tell whether $J_q(F)$ exists or not.

We will see below that $I_\lambda(F)$ and $J_q(F)$ are compositions of multiplication and convolution operators. In the lemma we study the appropriate convolution operators. We also use the following notation:

$$^{(y)}\!\int_{-\infty}^{\infty} f(u, y)du \equiv \underset{A\to\infty}{\text{l.i.m.}} \int_{-A}^{A} f(u, v)du$$

which means

$$\lim_{A\to\infty} \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty(y)} f(u, y)du - \int_{-A}^{A} f(u, y)du \right|^2 dy = 0.$$

2. LEMMA. (a) *Let*

$$(U_q\psi)(y) = (-iq/2\pi)^{1/2} \int_{-\infty}^{\infty(y)} \exp\left(\frac{qi(y-x)^2}{2}\right)\psi(x)dx$$

for $\psi \in L_2$, $y \in (-\infty, \infty)$ *and real* $q \neq 0$. U_q *is a unitary operator on* L_2 *and* $U_q^{\#} = U_q^{-1} = U_{-q}$ *where* $U_q^{\#}$ *denotes the adjoint of* U_q.

(b) *Let*

$$(C_\lambda\psi)(y) = (\lambda/2\pi)^{1/2} \int_{-\infty}^{\infty} \exp\left(\frac{-\lambda(y-x)^2}{2}\right)\psi(x)dx$$

for $\psi \in L_2$, $y \in (-\infty, \infty)$ *and* $\operatorname{Re}\lambda > 0$. C_λ *is in* $\mathfrak{L}(L_2)$, *it is one-to-one, its range is contained in the set of equivalence classes of* L_2 *which contain a continuous function, and* $\|C_\lambda\| = 1$.

PROOF. (a) The fact that U_q is an isometry in $\mathfrak{L}(L_2)$ is shown in Lemma 1 of [2], hence to show U_q unitary, we only need to show that it is onto. Now

$$(U_q\psi)(y) = (-iq/2\pi)^{1/2} \exp(iqy^2/2)\, \mathfrak{F}[\exp(iqx^2/2)\psi(x)](qy)$$

where $\mathfrak{F}$ denotes the L_2-Fourier transform. Thus to show U_q onto it suffices to show that, given an L_2 function $\phi(y)$, there exists an L_2 function ψ such that

$$\mathfrak{F}[\exp(iqx^2/2)\psi(x)](qy) = (2\pi/-iq)^{1/2} \exp(-iqy^2/2)\phi(y).$$

This follows since the maps $\psi(x) \to \exp(iqx^2/2)\psi(x)$ and $\mathfrak{F}$ are both onto L_2 and the q appearing in the argument just brings about a change of scale.

Since $U^{-1} = U^{\#}$ holds for any unitary operator, we need simply show $U_q^{\#} = U_{-q}$. Now the space C_K of continuous functions on $(-\infty, \infty)$ with compact support is dense in L_2 and so to show $U_q^{\#} = U_{-q}$ it suffices to show $(U_q\psi, \phi) = (\psi, U_{-q}\phi)$ for $\psi, \phi \in C_K$. This follows from the Fubini Theorem since the integrand in each case is dominated by the integrable function $|q|^{1/2}|\psi(x)||\phi(y)|$.

(b) C_λ is one-to-one for $(C_\lambda\psi)(y) = 0$ a.e. implies $\psi(x) = 0$ a.e. since the Fourier transform of a convolution is the product of the Fourier transforms and the Fourier transform of $\exp(-\lambda x^2/2)$ never vanishes. For any $\psi \in L_2$, $(C_\lambda\psi)(y)$ is continuous since the convolution of L_2 functions is continuous.

In Lemma 1 of [2], Cameron and Storvick show that $\|C_\lambda\| \leqq 1$. Using a comment in [3, p. 1045] one can show quite easily that $\|C_\lambda\| = 1$. However, the following elementary proof seems more instructive. Let $\frac{1}{2}<r<k<1$ be given. It suffices to find $\psi\in L_2$ such that $\|\psi\|=1$ and $\|C_\lambda\psi\|>r$. Let $\sigma^2=2|\lambda|^{-2}\mathrm{Re}\,\lambda$. Now choose integers m and n such that $(2\pi\sigma^2)^{-1/2}\int_{-m\sigma}^{m\sigma}\exp(-x^2/2\sigma^2)dx>k$ and $(n-m\sigma)k/n>r$. Let $\psi(x)=(2n)^{-1/2}\chi_{(-n,n)}(x)$. Then $\|\psi\|=1$, and by use of the formula

$$\int_{-\infty}^{\infty}\exp\left(\frac{-\lambda(x-\xi)^2}{2}\ \frac{-\bar{\lambda}(u-\xi)^2}{2}\right)d\xi$$
$$= (\pi/\mathrm{Re}\,\lambda)^{1/2}\exp\left(\frac{-|\lambda|^2(x-y)^2}{4\,\mathrm{Re}\,\lambda}\right).$$

We obtain

$$\begin{aligned}\|C_\lambda\psi\|^2 &= (2\pi\sigma^2)^{-1/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi(u)\psi(x)\exp\left(\frac{-(x-u)^2}{2\sigma^2}\right)dx\,du\\
&\geqq (2\pi\sigma^2)^{-1/2}(2n)^{-1}\int_{-n}^{n}\int_{u-n}^{u+n}\exp(-s^2/2\sigma^2)ds\,du\\
&\geqq (2n)^{-1}\int_{-n+m\sigma}^{n-m\sigma}\int_{u-n}^{u+n}(2\pi\sigma^2)^{-1/2}\exp(-s^2/2\sigma^2)ds\,du\\
&\geqq (2n)^{-1}\int_{-n+m\sigma}^{n-m\sigma}\int_{-m\sigma}^{m\sigma}(2\pi\sigma^2)^{-1/2}\exp(-s^2/2\sigma^2)ds\,du\\
&\geqq (2n-2m\sigma)k/2n > r > r^2.\end{aligned}$$

The following proposition establishes the existence of the operator $I_\lambda(F)$ for the class of F's we are considering.

PROPOSITION. *$I_\lambda(F)$ exists for all λ such that* $\mathrm{Re}\,\lambda>0$ *and is given by*

$$\begin{aligned}(I_\lambda(F)\psi)(\xi) &= \lambda^{n/2}[(2\pi)^n(t_1-a)\cdots(t_n-t_{n-1})]^{-1/2}\\
&\quad\cdot\int_{-\infty}^{\infty}(n)\cdot\int_{-\infty}^{\infty}\psi(v_n)\cdot f_1(v_1)\cdots f_n(v_n)\\
&\quad\cdot\exp\left(-\sum_{j=1}^{n}\frac{\lambda(v_j-v_{j-1})^2}{2(t_j-t_{j-1})}\right)dv_1\cdots dv_n\end{aligned}\tag{3}$$

where $v_0\equiv\xi$ and $t_0\equiv a$.

PROOF. Let $K_\lambda(F)$ denote the operator defined by the right-hand side of (3). In order to show that $I_\lambda^{\mathrm{seq}}(F)$ exists and equals $K_\lambda(F)$ it suffices to show that

$$\lim_{\text{norm}\,\sigma\to 0} (I_\lambda^\sigma(F)\psi, \phi) = (K_\lambda(F)\psi, \phi)$$

for all $\psi, \phi \in L_2$ where $I_\lambda^\sigma(F)$ is given by equation (2). This follows since for any partition $\sigma: a = s_0 < s_1 < \cdots < s_m = b$ with norm $\sigma < \min\{t_1 - a, \cdots, t_n - t_{n-1}\}$ we obtain

$$(I_\lambda^\sigma(F)\psi)(\xi) = \lambda^{n/2}[(2\pi)^n(r_1 - a)\cdots(r_n - r_{n-1})]^{-1/2}$$

$$\cdot \int_{-\infty}^{\infty}(n)\cdot\int_{-\infty}^{\infty} \psi(v_n)\cdot f_1(v_1)\cdots f_n(v_n)$$

$$\cdot \exp\left(-\sum_{j=1}^{n} \frac{\lambda(v_j - v_{j-1})^2}{2(r_j - r_{j-1})}\right) dv_1 \cdots dv_n$$

by carrying out $m - n$ integrations on the right side of equation (2), where $v_0 \equiv \xi$, $r_0 \equiv a$ and r_j is that s_k such that $s_k \leqq t_j < s_{k+1}$. Then as norm $\sigma \to 0$, $r_j \to t_j^-$ and so the result follows from the dominated convergence theorem.

For $\lambda > 0$, $K_\lambda(F)$ agrees with the Wiener integral given in equation (1) and is analytic for $\operatorname{Re}\lambda > 0$ [2, p. 533], and so $I_\lambda^{\text{an}}(F)$ exists and equals $K_\lambda(F)$.

THEOREM. *$J_q(F)$ exists for all real $q \neq 0$ and is given by the right-hand side of* (3) *where $\lambda = -iq$ and the integrals are interpreted in the mean. In fact $J_q(F)$ is the strong operator limit of $I_\lambda(F)$ as $\lambda \to -iq$ in the right-half plane.*

PROOF. We will establish that the operator defined by the right-hand side of (3) (with $\lambda = -iq$), which we will temporarily denote by $K_q(F)$, is the strong operator limit of $I_\lambda(F)$ as $\lambda \to -iq$ in the right half plane; the first statement then follows. Careful examination of $I_\lambda(F)$ reveals that it is the composition of multiplication operators and convolution operators [2, p. 535]; i.e. $I_\lambda(F) = C_{1,\lambda} \circ M_1 \circ \cdots \circ C_{n,\lambda} \circ M_n$ where $C_{j,\lambda}$ and M_j are the elements of $\mathfrak{L}(L_2)$ defined respectively by

$$(C_{j,\lambda}\psi)(y) = \left(\frac{\lambda}{2\pi(t_j - t_{j-1})}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left(\frac{-\lambda(y - x)^2}{2(t_j - t_{j-1})}\right) \psi(x)\, dx$$

and $(M_j\psi)(y) = \psi(y)f_j(y)$. Similarly $K_q(F) = U_1 \circ M_1 \circ \cdots \circ U_n \circ M_n$ where

$$(U_j\psi)(y) = \left(\frac{-iq}{2\pi(t_j - t_{j-1})}\right)^{1/2} {}^{(\nu)}\!\!\int_{-\infty}^{\infty} \exp\left(\frac{iq(y - x)^2}{2(t_j - t_{j-1})}\right) \psi(x) dx.$$

Now the map $(A, B) \to A \circ B$ of $L_2 \times L_2$ to L_2 is continuous in the strong operator topology as long as A is restricted to lie in a bounded subset of $\mathfrak{L}(L_2)$ [**3**, p. 512]. Since $\|C_{j,\lambda}\| = 1$ it will suffice to show that $C_{j,\lambda} \to U_j$(s.o.) where the s.o. refers to the fact that the convergence is with respect to the strong operator topology. This suffices since if $C_{n,\lambda} \to U_n$(s.o.) then $C_{n,\lambda} \circ M_n \to U_n \circ M_n$(s.o.) and $M_{n-1} \circ C_{n,\lambda} \circ M_n \to M_{n-1} \circ U_n \circ M_n$(s.o.) and $C_{n-1,\lambda} \circ M_{n-1} \circ C_{n,\lambda} \circ M_n \to U_{n-1} \circ M_{n-1} \circ U_n \circ M_n$(s.o.), etc. In showing $C_{j,\lambda} \to U_j$(s.o.) it clearly suffices to consider the case where $t_j - t_{j-1} = 1$. Hence, in our notation, $U_j = U_q$ and $C_{j,\lambda} = C_\lambda$. Now to show $C_\lambda \to U_q$(s.o.), it suffices to show $\|C_\lambda \psi - U_q \psi\| \to 0$ for $\psi \in C_K$ since for $\psi_0 \in L_2$ we have the inequalities

$$\begin{aligned} \|C_\lambda \psi_0 - U_q \psi_0\| &\leqq \|(C_\lambda - U_q)(\psi_0 - \psi)\| + \|C_\lambda \psi - U_q \psi\| \\ &\leqq 2\|\psi_0 - \psi\| + \|C_\lambda \psi - U_q \psi\|. \end{aligned} \tag{4}$$

Now to show $\|C_\lambda \psi - U_q \psi\| \to 0$, it suffices [**4**, p. 11] to show (a) $C_\lambda \psi \to U_q \psi$ weakly and (b) $\|C_\lambda \psi\| \to \|U_q \psi\|$. To obtain (a), it suffices to show $(C_\lambda \psi, \phi) \to (U_q \psi, \phi)$ for $\phi \in C_K$; one may see this by an inequality similar to (4). Since $(C_\lambda \psi, \phi)$ is a complex-valued function, it suffices to show $(C_{\lambda_m} \psi, \phi) \to (U_q \psi, \phi)$ along any sequence $\lambda_m \to -iq$ with $\operatorname{Re} \lambda_m > 0$. But as $\psi, \phi \in C_K$, $|\psi \phi|$ is integrable over $(-\infty, \infty) \times (-\infty, \infty)$ and so the result follows upon application of the Fubini Theorem and the dominated convergence theorem.

To establish (b) it again suffices to show $\|C_{\lambda_m} \psi\| \to \|U_q \psi\|$ for any sequence $\lambda_m \to -iq$ with $\operatorname{Re} \lambda_m > 0$. Now since $\|C_{\lambda_m}\| = 1$ and U_q is an isometry, we have $\liminf \|C_{\lambda_m} \psi\| \leqq \limsup \|C_{\lambda_m} \psi\| \leqq \|\psi\| = \|U_q \psi\|$. But also, since balls about zero in L_2 are compact in the weak topology, only finitely many of the C_{λ_m} can be in any such ball of radius less than $\|U_q \psi\|$. Hence $\|U_q \psi\| \leqq \liminf \|C_{\lambda_m} \psi\|$ and so $\lim \|C_{\lambda_m} \psi\| = \|U_q \psi\|$ which completes the proof of the theorem.

COROLLARY. *$J_q(F)$ is invertible as an element in $\mathfrak{L}(L_2)$ if and only if each f_j is bounded away from zero a.e. In this case $J_q(F)^{-1}$ is given by the formula*

$$\begin{aligned} (J_q(F)^{-1}\psi)(\xi) = {} & (iq)^{n/2}[(2\pi)^n (t_1 - a) \cdots (t_n - t_{n-1})]^{-1/2} \quad {}^{(\xi)} \\ & \cdot \int_{-\infty}^{\infty} (n) \cdot \overset{(v_{n-1})}{\cdots} \int_{-\infty}^{\infty} \psi(v_n) \\ & \cdot [f_n(\xi) f_{n-1}(v_1) f_{n-2}(v_2) \cdots f_1(v_{n-1})]^{-1} \\ & \cdot \exp\left(\frac{-iq}{2} \sum_{j=1}^{n} \frac{(v_{n-j+1} - v_{n-j})^2}{(t_j - t_{j-1})} \right) dv_n \cdots dv_1 \end{aligned} \tag{5}$$

where $v_0 \equiv \xi$ and $t_0 \equiv a$.

PROOF. It is easily verified [4, p. 32] that M_j is invertible if and only if f_j is bounded away from zero a.e., and, in this case, M_j^{-1} is multiplication by $1/f_j$. Thus under the conditions of the corollary, it follows from the lemma and theorem above that $J_q(F)^{-1} = M_n^{-1} \circ U_n^{-1} \circ \cdots \circ M_1^{-1} \circ U_1^{-1}$ exists and is given by (5). If $J_q(F)$ is not invertible, some M_j must fail to be invertible by the lemma and so f_j is not bounded away from zero a.e.

ACKNOWLEDGEMENT. The second author wishes to thank the University of Nebraska Research Council for financial support.

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