EXTENDING PARTIAL PERMUTATION MATRICES

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ABSTRACT. Let $M_1, M_2, \dots, M_s$ be $n \times n$ arrays such that in each M_i , each cell is either empty or occupied by a 1. It is shown that if $M_1 + M_2 + \dots + M_s$ contains only 1's, the totality of 1's is less than or equal to $n-1$, and the 1's are in different rows and columns, then the M_i 's can be completed to permutation matrices $M'_1, \dots, M'_s$ so that $M'_1 + \dots + M'_s$ is a $(0, 1)$ -matrix.

Let t be a positive integer, $1 \leq t \leq n-1$, and let $t = t_1 + t_2 + \dots + t_s$ be a partition of t . Let $M_1, M_2, \dots, M_s$ be $n \times n$ arrays such that M_i contains exactly t_i 1's on the main diagonal and the other cells are empty.

THEOREM. *If $M_1 + M_2 + \dots + M_s$ contains only 1's on the main diagonal, then the M_i 's can be completed to permutation matrices $M'_1, M'_2, \dots, M'_s$ such that $M'_1 + M'_2 + \dots + M'_s$ is a $(0, 1)$ -matrix.*

PROOF. Construct an $n \times n$ partial latin square P by placing an i in the t_i cells which are occupied in M_i . Then P is a partial latin square containing at most $n-1$ occupied cells, all on the main diagonal. J. Marica and J. Schönheim have shown that such a partial latin square can be completed [1]. Let P' be such a completion. Now define M'_i to be the permutation matrix containing a 1 in the cells occupied by i in P' and a 0 in the other cells. Then by construction M'_i is a completion of M_i and of course $M'_1 + M'_2 + \dots + M'_s$ is a $(0, 1)$ -matrix.

REMARK. If instead of the main diagonal the t occupied cells are in different rows and columns we obtain the same result.

REFERENCE

1. J. Marica and J. Schönheim, *Incomplete diagonals of latin squares*, Canad. Math. Bull. 12(1969), 235.

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