QUASI-PROJECTIVE COVERS AND DIRECT SUMS

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ABSTRACT. In this paper R denotes an associative ring with an identity, and all modules are unital left R-modules. It is shown that the existence of a quasi-projective cover for each module implies that each module has a projective cover. By a similar technique the following statements are shown to be equivalent: 1. R is semisimple and Artinian; 2. Every finitely generated module is quasi-projective; and 3. The direct sum of every pair of quasi-projective modules is quasi-projective. Direct sums of quasi-injective modules are also investigated.

1. Quasi-projective covers. Rings which have a projective cover for each module are called left perfect rings, and such rings have been characterized by H. Bass [1]. A module M has a projective cover P(M) iff there is an epimorphism $\phi: P(M) \rightarrow M$ such that P(M) is a projective module, and Ker ϕ is small in P(M) (i.e. A = P whenever $A + \text{Ker } \phi = P(M)$). A module is quasi-projective iff for every epimorphism $q: M \rightarrow A$, Hom(M, A) = q o Hom(M, M) [8] and [11]. Wu and Jans have defined a quasi-projective cover as follows: QP(M) is a quasi-projective cover of M iff there exists an epimorphism $\phi: QP(M) \rightarrow M$ such that 1. QP(M) is quasi-projective, 2. Ker ϕ is small in QP(M), and 3. if $0 \neq B \subseteq \text{Ker } \phi$, then QP(M)/B is not quasi-projective. They show that if a module has a projective cover, then it must have a quasi-projective cover. The converse to this theorem is false. The question of what can be said if each module has a quasi-projective cover is answered by the first theorem.

THEOREM 1.1. If every module has a quasi-projective cover, then every module has a projective cover.

PROOF. Let M be an arbitrary module and R^M be the direct sum of card (M) copies of R. There is an epimorphism $\phi: R^M \to M$. Let $\pi: Q \to R^M \oplus M$ be a quasi-projective cover of $R^M \oplus M$ and $q: R^M \oplus M \to R^M$ be the usual projection map. Since R^M is projective, there is a monomorphism $i: R^M \to Q$ such that $q \circ \pi \circ i = \text{identity on } R^M$, and $Q = \text{Im}(i) \oplus \text{Ker}(q \circ \pi)$. Let $\overline{M} = \text{Ker}(q \circ \pi)$ and $\pi_1 = \pi \mid_{\overline{M}}$. Then we can assume $Q = R^M \oplus \overline{M}$. We claim that \overline{M} is the projective cover of M with the desired epimorphism being π_1 . Suppose Ker $\pi_1 + A = \overline{M}$. Since Ker $\pi_1 \subseteq \text{Ker } \pi$ and $R^M \oplus \text{Ker } \pi_1 + A = R^M \oplus \overline{M} = Q$, we have

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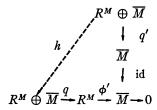
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 $R^M \oplus A = R^M \oplus \overline{M}$ or $A = \overline{M}$. Thus Ker π_1 is small in \overline{M} .

Now we need to show that \overline{M} is projective. Let $q': R^M \oplus \overline{M} \to \overline{M}$ be the projection onto \overline{M} and $j: \overline{M} \to R^M \oplus \overline{M}$ be the monomorphism such that $q' \circ j = \mathrm{id}$. R^M is projective, so there is a homomorphism $\phi': R^M \to \overline{M}$ such that

$$\begin{array}{c|c}
R^{M} \\
\phi' / & \phi \\
\hline
M & M \longrightarrow 0
\end{array}$$

commutes. The homomorphism ϕ' is onto because Ker π_1 is small. Since $R^M \oplus \overline{M}$ is quasi-projective there is an $h \in \operatorname{End}_R(R^M \oplus \overline{M})$ such that



commutes, that is, $\phi' \circ q \circ h = \mathrm{id} \circ q'$. Let $f = q \circ h \circ j$. Then $\phi' \circ f = \mathrm{id}$, and \overline{M} is isomorphic to a direct summand of R^M . Hence \overline{M} is the projective cover of M.

COROLLARY 1.2. Every module has a quasi-projective cover iff R is left perfect.

COROLLARY 1.3. A ring R is semiperfect [1] iff every finitely generated module has a quasi-projective cover.

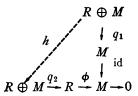
REMARK. It should be noted that property 3 in the definition of a quasi-projective cover was not needed for the proof of Theorem 1.1.

2. **Direct sums.** Direct sums of quasi-projective (quasi-injective) modules do not need to be quasi-projective (quasi-injective). Necessary and sufficient conditions for these direct sums to be quasi-projective (quasi-injective) will now be studied. A module M is quasi-injective iff for every monomorphism $i: A \rightarrow M$, $\operatorname{Hom}(A, M) = \operatorname{Hom}(M, M) \circ i$.

THEOREM 2.1. The following statements are equivalent:

- 1. R is semisimple and Artinian;
- 2. Every finitely generated module is quasi-projective; and
- 3. The direct sum of two quasi-projective modules is always quasi-projective.

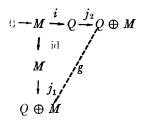
PROOF. It is known that 1 implies every module is projective. Thus 1 implies 2 and 3. It is also known that if every simple module is projective, then R is semisimple and Artinian. We will show that R is semisimple and Artinian if $R \oplus M$ is quasi-projective for every simple module M. It follows from this result that 2 implies 1 and 3 implies 1. Assume M is simple. Then there is an epimorphism $\phi: R \to M$. Let $q_1: R \oplus M \to M$ be the projection of $R \oplus M$ onto M and $q_2: R \oplus M \to R$ be the projection onto R. Let $i: M \to R \oplus M$ be the monomorphism such that q_1 o i=id. If $R \oplus M$ is quasi-injective then there is an $h \in \operatorname{End}(R \oplus M)$ such that



commutes. Let $f = q_2 \circ h \circ i$. Then $\phi \circ f = id$, and M is isomorphic to a direct summand of R. Thus every simple module is projective.

THEOREM 2.2. If the direct sum of two quasi-injective modules is always quasi-injective, then every quasi-injective module is injective.

PROOF. Let M be a quasi-injective module and Q be its injective hull. There is a monomorphism $i: M \rightarrow Q$. Let $j_1: M \rightarrow Q \oplus M$ be the injection of M into $Q \oplus M$ and $j_2: Q \rightarrow Q \oplus M$ be the injection of Q. Let $q: Q \oplus M \rightarrow M$ be the epimorphism such that $q \circ j_1 = \text{id}$. Since $Q \oplus M$ is quasi-injective, there is a $g \in \text{End}(Q \oplus M)$ such that



commutes. Let $f = q \circ g \circ j_2$. Then $f \circ i = id$, and hence, M is isomorphic to a direct summand of Q. Therefore M is injective.

COROLLARY 2.3. If the direct sum of every two quasi-injective modules is quasi-injective, then R is semisimple and Noetherian.

¹ This result was proved by G. Azumaya for a class in 1965 but is unpublished. See [10] for a published proof that R is semisimple if every maximal right ideal is a direct summand of R.

PROOF. Since completely reducible modules are quasi-injective, they are injective. In particular every simple module is injective. Beachy has shown in this case that every proper ideal is the intersection of maximal ideals. Thus R is semisimple. Kurshan has proved that if direct sums of injective hulls of simple modules are injective, then R is Noetherian [6]. Therefore, R is Noetherian in the present case.

COROLLARY 2.4. A ring R is self-injective, and the direct sum of every pair of quasi-injective modules is quasi-injective iff R is semisimple and Artinian.

PROOF. It is known that if R is injective and Noetherian, then R is Artinian [4]. So by Corollary 2.3 the "only if" direction is proved. The converse is true because every module is injective if R is Artinian and semisimple.

Remarks. 1. Corollary 2.4 strengthens a result of Chaptal [2].

2. If the conjecture that a ring is von Neumann regular if every simple module is injective is true, then one would not need to assume R is injective in Corollary 2.4. This result is true when R is commutative [9].

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