QUASI-PROJECTIVE COVERS AND DIRECT SUMS

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ABSTRACT. In this paper R denotes an associative ring with an identity, and all modules are unital left R-modules. It is shown that the existence of a quasi-projective cover for each module implies that each module has a projective cover. By a similar technique the following statements are shown to be equivalent: 1. R is semisimple and Artinian; 2. Every finitely generated module is quasi-projective; and 3. The direct sum of every pair of quasi-projective modules is quasi-projective. Direct sums of quasi-injective modules are also investigated.

1. **Quasi-projective covers.** Rings which have a projective cover for each module are called *left perfect* rings, and such rings have been characterized by H. Bass [1]. A module M has a *projective cover* $P(M)$ iff there is an epimorphism $\phi: P(M) \to M$ such that $P(M)$ is a projective module, and Ker ϕ is small in $P(M)$ (i.e. $A = P$ whenever $A + \text{Ker } \phi = P(M)$). A module is *quasi-projective* iff for every epimorphism $q: M \to A$, $\text{Hom}(M, A) = q \circ \text{Hom}(M, M)$ [8] and [11]. Wu and Jans have defined a *quasi-projective cover* as follows: $QP(M)$ is a quasi-projective cover of M iff there exists an epimorphism $\phi: QP(M) \to M$ such that 1. $QP(M)$ is quasi-projective, 2. Ker ϕ is small in $QP(M)$, and 3. if $0 \neq B \subseteq \text{Ker } \phi$, then $QP(M)/B$ is not quasi-projective. They show that if a module has a projective cover, then it must have a quasi-projective cover. The converse to this theorem is false. The question of what can be said if each module has a quasi-projective cover is answered by the first theorem.

THEOREM 1.1. *If every module has a quasi-projective cover, then every module has a projective cover.*

PROOF. Let M be an arbitrary module and R^M be the direct sum of card(M) copies of R. There is an epimorphism $\phi: R^M \to M$. Let $\pi: Q \to R^M \oplus M$ be a quasi-projective cover of $R^M \oplus M$ and $q: R^M \oplus M \to R^M$ be the usual projection map. Since R^M is projective, there is a monomorphism $i: R^M \to Q$ such that $q \circ \pi \circ i = $ identity on R^M, and $Q = \text{Im}(i) \oplus \text{Ker}(q \circ \pi)$. Let $\overline{M} = \text{Ker}(q \circ \pi)$ and $\pi_1 = \pi|_{\overline{M}}$. Then we can assume $Q = R^M \oplus \overline{M}$. We claim that $\overline{M}$ is the projective cover of M with the desired epimorphism being π_1. Suppose Ker $\pi_1 + A = \overline{M}$. Since Ker $\pi_1 \subseteq$ Ker π and $R^M \oplus \text{Ker } \pi_1 + A = R^M \oplus \overline{M} = Q$, we have

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$R^M \oplus A = R^M \oplus \overline{M}$ or $A = \overline{M}$. Thus Ker π_1 is small in $\overline{M}$.

Now we need to show that $\overline{M}$ is projective. Let $q': R^M \oplus \overline{M} \to \overline{M}$ be the projection onto $\overline{M}$ and $j: \overline{M} \to R^M \oplus \overline{M}$ be the monomorphism such that $q' \circ j = \mathrm{id}$. R^M is projective, so there is a homomorphism $\phi': R^M \to \overline{M}$ such that

$$\begin{tikzcd}
 & R^M \arrow[dl, dashed, "\phi'"'] \arrow[d, "\phi"] & \\
\overline{M} \arrow[r, "\pi_1"] & M \arrow[r] & 0
\end{tikzcd}$$

commutes. The homomorphism ϕ' is onto because Ker π_1 is small. Since $R^M \oplus \overline{M}$ is quasi-projective there is an $h \in \mathrm{End}_R(R^M \oplus \overline{M})$ such that

$$\begin{tikzcd}
 & & R^M \oplus \overline{M} \arrow[dll, dashed, "h"'] \arrow[d, "q'"] & \\
 & & \overline{M} \arrow[d, "\mathrm{id}"] & \\
R^M \oplus \overline{M} \arrow[r, "q"] & R^M \arrow[r, "\phi'"] & \overline{M} \arrow[r] & 0
\end{tikzcd}$$

commutes, that is, $\phi' \circ q \circ h = \mathrm{id} \circ q'$. Let $f = q \circ h \circ j$. Then $\phi' \circ f = \mathrm{id}$, and $\overline{M}$ is isomorphic to a direct summand of R^M. Hence $\overline{M}$ is the projective cover of M.

COROLLARY 1.2. *Every module has a quasi-projective cover iff R is left perfect.*

COROLLARY 1.3. *A ring R is semiperfect* [1] *iff every finitely generated module has a quasi-projective cover.*

REMARK. It should be noted that property 3 in the definition of a quasi-projective cover was not needed for the proof of Theorem 1.1.

2. **Direct sums.** Direct sums of quasi-projective (quasi-injective) modules do not need to be quasi-projective (quasi-injective). Necessary and sufficient conditions for these direct sums to be quasi-projective (quasi-injective) will now be studied. A module M is quasi-injective iff for every monomorphism $i: A \to M$, $\mathrm{Hom}(A, M) = \mathrm{Hom}(M, M) \circ i$.

THEOREM 2.1. *The following statements are equivalent:*

1. *R is semisimple and Artinian;*
2. *Every finitely generated module is quasi-projective; and*
3. *The direct sum of two quasi-projective modules is always quasi-projective.*

PROOF. It is known that 1 implies every module is projective. Thus 1 implies 2 and 3. It is also known that if every simple module is projective, then R is semisimple and Artinian.[1] We will show that R is semisimple and Artinian if $R\oplus M$ is quasi-projective for every simple module M. It follows from this result that 2 implies 1 and 3 implies 1. Assume M is simple. Then there is an epimorphism $\phi: R \to M$. Let $q_1: R\oplus M \to M$ be the projection of $R\oplus M$ onto M and $q_2: R\oplus M\to R$ be the projection onto R. Let $i: M\to R\oplus M$ be the monomorphism such that $q_1 \circ i = \mathrm{id}$. If $R\oplus M$ is quasi-injective then then there is an $h \in \mathrm{End}(R\oplus M)$ such that

$$\begin{tikzcd}
 & & & R \oplus M \arrow[d, "q_1"] \arrow[dddlll, dashed, "h"'] \\
 & & & M \arrow[dd, "\mathrm{id}"] \\
 & & & \\
R \oplus M \arrow[r, "q_2"] & R \arrow[rr, "\phi"] & & M \arrow[r] & 0
\end{tikzcd}$$

commutes. Let $f = q_2 \circ h \circ i$. Then $\phi \circ f = \mathrm{id}$, and M is isomorphic to a direct summand of R. Thus every simple module is projective.

THEOREM 2.2. *If the direct sum of two quasi-injective modules is always quasi-injective, then every quasi-injective module is injective.*

PROOF. Let M be a quasi-injective module and Q be its injective hull. There is a monomorphism $i: M\to Q$. Let $j_1: M\to Q\oplus M$ be the injection of M into $Q\oplus M$ and $j_2: Q\to Q\oplus M$ be the injection of Q. Let $q: Q\oplus M\to M$ be the epimorphism such that $q \circ j_1 = \mathrm{id}$. Since $Q\oplus M$ is quasi-injective, there is a $g\in \mathrm{End}(Q\oplus M)$ such that

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "i"] \arrow[d, "\mathrm{id}"] & Q \arrow[r, "j_2"] & Q \oplus M \arrow[ddll, dashed, "g"] \\
 & M \arrow[d, "j_1"] & & \\
 & Q \oplus M & &
\end{tikzcd}$$

commutes. Let $f = q \circ g \circ j_2$. Then $f \circ i = \mathrm{id}$, and hence, M is isomorphic to a direct summand of Q. Therefore M is injective.

COROLLARY 2.3. *If the direct sum of every two quasi-injective modules is quasi-injective, then R is semisimple and Noetherian.*

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[1] This result was proved by G. Azumaya for a class in 1965 but is unpublished. See [10] for a published proof that R is semisimple if every maximal right ideal is a direct summand of R.

PROOF. Since completely reducible modules are quasi-injective, they are injective. In particular every simple module is injective. Beachy has shown in this case that every proper ideal is the intersection of maximal ideals. Thus R is semisimple. Kurshan has proved that if direct sums of injective hulls of simple modules are injective, then R is Noetherian [6]. Therefore, R is Noetherian in the present case.

COROLLARY 2.4. *A ring R is self-injective, and the direct sum of every pair of quasi-injective modules is quasi-injective iff R is semisimple and Artinian.*

PROOF. It is known that if R is injective and Noetherian, then R is Artinian [4]. So by Corollary 2.3 the "only if" direction is proved. The converse is true because every module is injective if R is Artinian and semisimple.

REMARKS. 1. Corollary 2.4 strengthens a result of Chaptal [2].

2. If the conjecture that a ring is von Neumann regular if every simple module is injective is true, then one would not need to assume R is injective in Corollary 2.4. This result is true when R is commutative [9].

REFERENCES

1. H. Bass, *Finitistic dimension and a homological generalization of semi-primary rings*, Trans. Amer. Math. Soc. **95** (1960), 466–488. MR **28** #1212.

2. N. Chaptal, *Sur les modules quasi-injectifs*, C. R. Acad. Sci. Paris Sér. A-B **264** (1967), A173–A175. MR **34** #7574.

3. J. Beachy, *Some homological classes of rings and modules*, Ph.D. Thesis, Indiana University, Bloomington, Ind., 1967.

4. C. Faith *Rings with ascending condition on annihilators*, Nagoya Math. J. **27** (1966), 179–191. MR **33** #1328.

5. C. Faith and Y. Utumi, *Quasi-injective modules and their endomorphism rings*, Arch. Math. **15** (1964), 166–174. MR **29** #3503.

6. R. P. Kurshan, *Rings whose cyclic modules have finitely generated socle*, (to appear).

7. J. Lambek, *Lectures on rings and modules*, Blaisdell, Waltham, Mass., 1966. MR **34** #5857.

8. Y. Miyashita, *Quasi-projective modules, perfect modules, and a theorem for modular lattices*, J. Fac. Sci. Hokkaido Univ. Ser I **19** (1966), 86–110. MR **35** #4254.

9. A. Rosenberg and D. Zelinsky, *Finiteness of the injective hull*, Math. Z. **70** (1958/59), 372–380. MR **21** #4176.

10. M. Satyanarayana, *Semisimple rings*, Amer. Math. Monthly **74** (1967), 1086. MR **37** #1403.

11. L. E. T. Wu and J. P. Jans, *On quasi projectives*, Illinois J. Math. **11** (1967), 439–448. MR **36** #3817.

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