ADDENDUM TO: SOME INTEGRAL CHARACTERIZATIONS OF ABSOLUTE CONTINUITY

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1. **Introduction.** Suppose U is a set, F is a field of subsets of U, $\mathfrak{p}$ is the set of all real-valued functions defined on F, $\mathfrak{p}_B$ is the set of all bounded elements of $\mathfrak{p}$, $\mathfrak{p}^+$ is the set of all nonnegative-valued elements of $\mathfrak{p}$, $\mathfrak{p}_B^+ = \mathfrak{p}_B \cap \mathfrak{p}^+$, $\mathfrak{p}_A$ is the set of all bounded finitely additive elements of $\mathfrak{p}$, and $\mathfrak{p}_A^+ = \mathfrak{p}_A \cap \mathfrak{p}^+$. Suppose furthermore that for each μ in $\mathfrak{p}_A$, $\mathfrak{A}_\mu$ is the set of all ξ in $\mathfrak{p}_A$ absolutely continuous with respect to μ, and $\mathfrak{I}_\mu$ is the set of all α in $\mathfrak{p}_B$ such that the integral (§2)

$$\int_U \alpha(I)\mu(I)$$

exists. Finally, suppose $\mathfrak{Z}$ is the set of all β in $\mathfrak{p}$ such that for each I in F, $\beta(I)$ is 1 or 0.

In a previous paper [1] the author demonstrated a theorem, part of which is the following equivalence assertion:

THEOREM 1.A.1. *If each of μ and ξ is in $\mathfrak{p}_A^+$, then the following two statements are equivalent*:

(1) *If β is in $\mathfrak{Z} \cap \mathfrak{I}_\mu \cap \mathfrak{I}_\xi$ and $\int_U \beta(I)\mu(I) = 0$, then $\int_U \beta(I)\xi(I) = 0$.*

(2) *ξ is in $\mathfrak{A}_\mu$.*

In the same paper the author gave the following characterization theorem:

THEOREM 1.A.2. *The following two statements are equivalent*:

(1) *If η is in $\mathfrak{p}_A^+$ and $\eta(U) > 0$, then there is some element of $\mathfrak{p}_B^+$ not in $\mathfrak{I}_\eta$.*

(2) *If each of μ and ξ is in $\mathfrak{p}_A^+$, then ξ is in $\mathfrak{A}_\mu$ iff $\mathfrak{p}_B^+ \cap \mathfrak{I}_\mu \subseteq \mathfrak{p}_B^+ \cap \mathfrak{I}_\xi$ (which the reader can easily see is true iff $\mathfrak{I}_\mu \subseteq \mathfrak{I}_\xi$).*

In proving that in Theorem 1.A.2, (1) implies (2), it was shown, without using (1), that if each of μ and ξ is in $\mathfrak{p}_A^+$ and ξ is in $\mathfrak{A}_\mu$, then $\mathfrak{p}_B^+ \cap \mathfrak{I}_\mu \subseteq \mathfrak{p}_B^+ \cap \mathfrak{I}_\xi$ (and hence that $\mathfrak{I}_\mu \subseteq \mathfrak{I}_\xi$); furthermore, in proving that (2) implies (1), the only part of (2) used was the statement that $\mathfrak{p}_B^+ \cap \mathfrak{I}_\mu \subseteq \mathfrak{p}_B^+ \cap \mathfrak{I}_\xi$ implies that ξ is in $\mathfrak{A}_\mu$. We can therefore give the following more specific version of Theorem 1.A.2:

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THEOREM 1.A.3. *The following two statements are true:*

(1) *If each of* μ *and* ξ *is in* $\mathfrak{p}_A^+$ *and* ξ *is in* $\mathfrak{A}_\mu$, *then* $\mathcal{G}_\mu \subseteq \mathcal{G}_\xi$.

(2) *The following two statements are equivalent:*

(i) *If* η *is in* $\mathfrak{p}_A^+$ *and* $\eta(U) > 0$, *then there is some element of* $\mathfrak{p}_B^+$ *not in* $\mathcal{G}_\eta$.

(ii) *If each of* μ *and* ξ *is in* $\mathfrak{p}_A^+$ *and* $\mathcal{G}_\mu \subseteq \mathcal{G}_\xi$, *then* ξ *is in* $\mathfrak{A}_\mu$.

Theorems 1.A.1 and 1.A.3 therefore tell us that absolute continuity has an integral characterization in terms of $\mathcal{Z}$, that absolute continuity implies a certain integrability set inclusion property, and that the only circumstance under which integrability set inclusion fails to imply absolute continuity is when a certain nonintegrability assertion fails to hold. Therefore the question naturally arises as to whether integrability set inclusion has an integral characterization in terms of $\mathcal{Z}$. In this paper we prove the following characterization theorem (§3):

THEOREM 3.1. *If each of* μ *and* ξ *is in* $\mathfrak{p}_A^+$, *then the following two statements are equivalent:*

(1) *If* β *is in* $\mathcal{Z} \cap \mathcal{G}_\mu$ *and* $\int_U \beta(I)\mu(I) = 0$, *then* β *is in* $\mathcal{G}_\xi$;

(2) $\mathcal{G}_\mu \subseteq \mathcal{G}_\xi$.

2. **Preliminary theorems and definitions.** We refer the reader to §§2 and 3 of [1] for some of the basic theorems and definitions used in this paper, and when the existence of an integral or its equivalence to an integral is an easy consequence of the above mentioned material, the integral need only be written, and the proof of existence or equivalence left to the reader.

We close this section by referring the reader to §2 of another paper of the author [2] for notions pertaining to Σ-boundedness and upper and lower integral, as well as pertinent basic facts, conventions and notation.

3. **The integrability inclusion characterization theorem.** We begin this section by stating a lemma about "sum supremum" and "sum infimum" functionals defined in [2] that the reader can easily prove.

LEMMA 3.1. *If* α *is in* $\mathfrak{p}$ *and is* Σ*-bounded with respect to the subdivision* $\mathfrak{D}$ *of* U *and* $0 \leqq c$, *then* $cs^*(\alpha) = s^*(c\alpha)$ *and* $cs_*(\alpha) = s_*(c\alpha)$.

We now prove Theorem 3.1, as stated in the introduction.

PROOF OF THEOREM 3.1. It is obvious that (2) implies (1).

Suppose (1) is true, but that for some γ in $\mathfrak{p}_B$, γ is in $\mathcal{G}_\mu$ and not in $\mathcal{G}_\xi \cdot \xi(U) > 0$. We adopt the convention that if each of a and b is a number, then $a/b = 0$ if $b = 0$, and has the usual meaning otherwise.

There is a number $M>0$ such that $|\gamma(I)| \leqq M$ for all I in $\boldsymbol{F}$. Letting $\alpha=\gamma+M$, we see that α is in $\mathfrak{p}_B^+ \cap \mathcal{I}_\mu$ and not in $\mathcal{I}_\xi$, and that $\alpha(I) \leqq 2M$ for all I in $\boldsymbol{F}$.

Let η be the element of $\mathfrak{p}_A^+$ defined by

$$\eta(V) = \int_V \max\{\mu(I), \xi(I)\}.$$

$\eta(U)>0$. We easily see that ξ/η is in $\mathfrak{p}_B^+ \cap \mathcal{I}_\eta$. If α is in $\mathcal{I}_\eta$, then it follows from Theorem 2.A.1 of [1] that $(\alpha)(\xi/\eta)$ is in $\mathcal{I}_\eta$, and since

$$\int_U \alpha(I)[\xi(I)/\eta(I)]\eta(I) = \int_U \alpha(I)\xi(I),$$

it follows that α is in $\mathcal{I}_\xi$, a contradiction. Therefore α is not in $\mathcal{I}_\eta$.

Now, let $\delta=\alpha-[s_*(\alpha\eta)]/\eta$. Obviously δ is in $\mathfrak{p}_B^+$. Since $[s_*(\alpha\eta)]/\eta$ is clearly in $\mathcal{I}_\eta$ and α is not, it follows that δ is not in $\mathcal{I}_\eta$. Therefore

$$\int_U s_*(\delta\eta)(I) < \int_U s^*(\delta\eta)(I).$$

We now show that $s_*(\delta\eta)(V)=0$ for all V in $\boldsymbol{F}$. Suppose V is in $\boldsymbol{F}$ and $0<c$. Obviously $0 \leqq s_*(\delta\eta)(V)$. There is a subdivision $\mathfrak{D}$ of V such that

$$0 \leqq \left[\sum_{\mathfrak{D}} \alpha(I)\eta(I)\right] - s_*(\alpha\eta)(V) < c,$$

so that

$$\begin{aligned} s_*(\delta\eta)(V) &\leqq \sum_{\mathfrak{D}} [\alpha(I)\eta(I) - s_*(\alpha\eta)(I)] \\ &\leqq \left[\sum_{\mathfrak{D}} \alpha(I)\eta(I)\right] - s_*(\alpha\eta)(V) < c. \end{aligned}$$

Therefore $s_*(\delta\eta)(V)=0$ for all V in $\boldsymbol{F}$, so that

$$0 = \int_U s_*(\delta\eta)(I).$$

We now show that δ is in $\mathcal{I}_\mu$ and $\int_U \delta(I)\mu(I)=0$. Since each of μ/η and $[s_*(\alpha\eta)]/\eta$ is in $\mathcal{I}_\eta$, it follows, again from Theorem 2.A.1 of [1], that $(\mu/\eta)([s_*(\alpha\eta)]/\eta)$ is in $\mathcal{I}_\eta$, and, since

$$\int_U [\mu(I)/\eta(I)][\{s_*(\alpha\eta)(I)\}/\eta(I)]\eta(I) = \int_U [\{s_*(\alpha\eta)(I)\}/\eta(I)]\mu(I),$$

it follows that $[s_*(\alpha\eta)]/\eta$ is in $\mathscr{I}_\mu$. Since α is in $\mathscr{I}_\mu$, it follows that δ is in $\mathscr{I}_\mu$. Now, suppose $0<c$. From a theorem of Kolmogoroff [**3**] (see also §2 of [**1**]) it follows that there is a subdivision $\mathfrak{D}$ of U such that if $\mathfrak{E}$ is a refinement of $\mathfrak{D}$, then

$$\sum_{\mathfrak{E}} \left| \alpha(I)\mu(I) - \int_I \alpha(J)\mu(J) \right| < c/4$$

and

$$\sum_{\mathfrak{E}} \left| [\{s_*(\alpha\eta)(I)\}/\eta(I)]\mu(I) - \int_I [\{s_*(\alpha\eta)(J)\}/\eta(J)]\mu(J) \right| < c/4.$$

Now, for each I' in $\mathfrak{D}$ there is a subdivision $\mathfrak{D}_{I'}$ of I' such that

$$\left[\sum_{\mathfrak{D}_{I'}} \alpha(J)\eta(J) \right] - s_*(\alpha\eta)(I') < c/(4N),$$

where N is the number of elements of $\mathfrak{D}$, so that

$$\begin{aligned} 0 &\leqq \sum_{\mathfrak{D}_{I'}} [\alpha(J)\eta(J) - s_*(\alpha\eta)(J)] \\ &\leqq \left[\sum_{\mathfrak{D}_{I'}} \alpha(J)\eta(J) \right] - s_*(\alpha\eta)(I') < c/(4N). \end{aligned}$$

Now,

$$\begin{aligned} 0 \leqq \int_U \delta(I)\mu(I) \leqq \sum_{\mathfrak{D}} \sum_{\mathfrak{D}_{I'}} \Bigg[&\left| \int_J \alpha(J')\mu(J') - \alpha(J)\mu(J) \right| \\ &+ | \alpha(J)\mu(J) - [\{s_*(\alpha\eta)(J)\}/\eta(J)]\mu(J) | + \Bigg| [\{s_*(\alpha\eta)(J)\}/\eta(J)]\mu(J) \\ &\qquad - \int_J [\{s_*(\alpha\eta)(J')\}/\eta(J')]\mu(J') \Bigg| \Bigg] < c/4 \\ &+ \left[\sum_{\mathfrak{D}} \sum_{\mathfrak{D}_{I'}} | \alpha(J) - [\{s_*(\alpha\eta)(J)\}/\eta(J)] | \, \mu(J) \right] + c/4 \\ \leqq c/4 + &\left[\sum_{\mathfrak{D}} \sum_{\mathfrak{D}_{I'}} | \alpha(J) - [\{s_*(\alpha\eta)(J)\}/\eta(J)] | \, \eta(J) \right] + c/4 \\ = c/4 + &\left[\sum_{\mathfrak{D}} \sum_{\mathfrak{D}_{I'}} \{\alpha(J)\eta(J) - s_*(\alpha\eta)(J)\} \right] + c/4 \\ < c/4 + &Nc/(4N) + c/4 = 3c/4 < c. \end{aligned}$$

Therefore $\int_U \delta(I)\mu(I)=0$.

Since δ is in $\mathfrak{p}_B^+$ and not in $\mathcal{I}_\eta$, it follows that there is a $t>0$ such that if $\mathfrak{D}$ is a subdivision of U, then there is a refinement $\mathfrak{E}$ of $\mathfrak{D}$ such that $t<\sum_{\mathfrak{E}} \delta(I)\eta(I)$. Let λ be the element of $\mathfrak{p}_B^+$ defined by

$$\begin{aligned} \lambda(I) &= t/[2\eta(U)] \qquad \text{if } \delta(I) \geqq t/[2\eta(U)], \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Since each of λ and $\delta-\lambda$ is in $\mathfrak{p}_B^+$, it follows that

$$\int_U \lambda(I)\mu(I) = 0 = \int_U \mathrm{s}_*(\lambda\eta)(I).$$

Let ω denote $(4M\eta(U)/t)\lambda$. $\int_U \omega(I)\mu(I)=0$, and from Lemma 3.1 it follows that $\int_U \mathrm{s}_*(\omega\eta)(I)=0$. Furthermore,

$$\begin{aligned} \omega(I) &= 2M \qquad \text{if } \delta(I) \geqq t/[2\eta(U)], \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Now, suppose $\mathfrak{D}$ is a subdivision of U. There is a refinement $\mathfrak{E}$ of $\mathfrak{D}$ such that $t<\sum_{\mathfrak{E}}\delta(I)\eta(I)$. Obviously there is an I in $\mathfrak{E}$ such that $\delta(I) \geqq t/[2\eta(U)]$. Letting $\mathfrak{E}^*=\{I \mid I \text{ in } \mathfrak{E},\ \delta(I)\geqq t/[2\eta(U)]\}$, we have

$$t < \sum_{\mathfrak{E}} \delta(I)\eta(I) \leqq t/2 + \sum_{\mathfrak{E}^*} \delta(I)\eta(I),$$

so that

$$\begin{aligned} t/2 &< \sum_{\mathfrak{E}^*} \delta(I)\eta(I) \leqq \sum_{\mathfrak{E}^*} 2M\eta(I) \\ &= \sum_{\mathfrak{E}^*} \omega(I)\eta(I) \leqq \sum_{\mathfrak{E}} \omega(I)\eta(I) \leqq \sum_{\mathfrak{E}} \mathrm{s}^*(\omega\eta)(I). \end{aligned}$$

Therefore

$$\int_U \mathrm{s}_*(\omega\eta)(I) = 0 < t/2 \leqq \int_U \mathrm{s}^*(\omega\eta)(I),$$

so that ω is not in $\mathcal{I}_\eta$.

Let $\sigma=(1/[2M])\omega$. We easily see that σ is in $\mathfrak{Z}\cap\mathcal{I}_\mu$ and $\int_U \sigma(I)\mu(I)=0$. We also see from Lemma 3.1 that

$$\int_U \mathrm{s}_*(\sigma\eta)(I) = 0 < t/[4M] \leqq \int_U \mathrm{s}^*(\sigma\eta)(I),$$

so that σ is not in $\mathcal{I}_\eta$. If σ is in $\mathcal{I}_\xi$, then

$$\int_U \max\left\{\int_I \sigma(J)\mu(J),\ \int_I \sigma(J)\xi(J)\right\}$$
$$= \int_U \max\{\sigma(I)\mu(I),\ \sigma(I)\xi(I)\}$$
$$= \int_U \sigma(I)\int_I \max\{\mu(J),\xi(J)\} = \int_U \sigma(I)\eta(I),$$

so that σ is in $\mathcal{G}_\eta$, a contradiction. Therefore σ is not in $\mathcal{G}_\xi$, a contradiction.

Therefore (1) implies (2).

Therefore (1) and (2) are equivalent.

References

1. W. D. L. Appling, *Some integral characterizations of absolute continuity*, Proc. Amer. Math. Soc. **18** (1967), 94–99. MR **34** #4446.

2. ———, *Integral representations and refinement-unboundedness*, Proc. Amer. Math. Soc. **19** (1968), 837–842. MR **37** #4227.

3. A. Kolmogoroff, *Untersuchen über den Integralbegriff*, Math. Ann. **103** (1930), 654–696.

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