AN m-ORTHOCOMPLETE ORTHOMODULAR LATTICE IS m-COMPLETE

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ABSTRACT. We call an orthomodular lattice $\mathfrak{L}$ m-orthocomplete for an infinite cardinal m if every orthogonal family of $\leqq m$ elements from $\mathfrak{L}$ has a join in $\mathfrak{L}$, and we call $\mathfrak{L}$ m-complete if every family, orthogonal or not, of $\leqq m$ elements from $\mathfrak{L}$ has a join in $\mathfrak{L}$. We prove that an m-orthocomplete orthomodular lattice is m-complete. Since a Boolean algebra is a distributive orthomodular lattice, we obtain as a special case the Smith-Tarski theorem: An m-orthocomplete Boolean algebra is m-complete.

We refer the reader to [1] for the elementary theory and basic nomenclature of orthomodular lattices, mentioning specifically here only these notational conventions: we write $a-b$ for $a\wedge b^{\perp}$ when $b\leqq a$, and write $\bigoplus a_\alpha$ for $\bigvee a_\alpha$ when $\alpha\neq\beta\Rightarrow a_\alpha\perp a_\beta$.

LEMMA. *Let $\mathfrak{L}$ be an m-orthocomplete orthomodular lattice, σ an ordinal number satisfying* $\operatorname{card}(\sigma)\leqq m$, *and $(y_\alpha;\ \alpha<\sigma)$ a family of elements from L satisfying*

(i) $y_0=0$,

(ii) $\alpha\leqq\beta<\sigma\Rightarrow y_\alpha\leqq y_\beta$ *(increasing)*,

(iii) *β a limit ordinal $<\sigma\Rightarrow\bigvee(y_\alpha;\alpha<\beta)$ exists and $=y_\beta$ (continuous from the left).*

Then for every ordinal β satisfying $2\leqq\beta<\sigma$ we have

$$\bigvee(y_\alpha;\alpha<\beta)=\bigoplus(y_{\rho+1}-y_\rho;\rho+1<\beta).$$

PROOF OF THE LEMMA. Both joins displayed in the assertion of the lemma exist, the orthogonal join by m-orthocompleteness, and the other by assumption (iii). (Assumption (iii) covers the case when β is a limit ordinal; if β is not a limit ordinal, then obviously $\bigvee(y_\alpha;\alpha<\beta)=y_{\beta-1}$.) If $\rho+1<\beta$, then $y_{\rho+1}-y_\rho\leqq y_{\rho+1}\leqq\bigvee(y_\alpha;\ \alpha<\beta)$; hence

$$\bigoplus(y_{\rho+1}-y_\rho;\rho+1<\beta)\leqq\bigvee(y_\alpha;\alpha<\beta).$$

We need therefore prove only the statement $P(\beta)$: $\bigvee(y_\alpha;\ \alpha<\beta)\leqq\bigoplus(y_{\rho+1}-y_\rho;\ \rho+1<\beta)$. $P(2)$ is the assertion $y_1\leqq y_1-y_0$ which is true because $y_0=0$. We use transfinite induction. Assume that $P(\gamma)$

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is true for all $\gamma<\beta$. If β is a limit ordinal, then for any $\alpha<\beta$, $\alpha+1<\beta$ and then, using the induction hypothesis,

$$\begin{aligned} y_\alpha = \bigvee(y_\sigma; \sigma \leqq \alpha) &= \bigvee(y_\sigma; \sigma < \alpha+1) \\ \leqq \bigoplus(y_{\rho+1}-y_\rho; \rho+1<\alpha+1) &\leqq \bigoplus(y_{\rho+1}-y_\rho; \rho+1<\beta). \end{aligned}$$

Hence $\bigvee(y_\alpha; \alpha<\beta) \leqq \bigoplus(y_{\rho+1}-y_\rho; \rho+1<\beta)$. If β is not a limit ordinal, then $\bigvee(y_\alpha;\ \alpha<\beta)=\bigvee(y_\alpha;\ \alpha\leqq\beta-1)=y_{\beta-1}$. Now there are two possibilities: either $\beta-1$ is a limit ordinal or it is not. If $\beta-1$ is a limit ordinal, then by (iii) and the induction hypothesis,

$$\begin{aligned} y_{\beta-1} = \bigvee(y_\alpha; \alpha<\beta-1) &\leqq \bigoplus(y_{\rho+1}-y_\rho; \rho+1<\beta-1) \\ &\leqq \bigoplus(y_{\rho+1}-y_\rho; \rho+1<\beta) \end{aligned}$$

and we are done. If $\beta-1$ is not a limit ordinal, then

$$\begin{aligned} &\bigoplus(y_{\rho+1}-y_\rho; \rho+1<\beta) \\ &\qquad = \bigoplus(y_{\rho+1}-y_\rho; \rho+1\leqq\beta-1) \\ &\qquad = (y_{\beta-1}-y_{\beta-2}) \oplus \bigoplus(y_{\rho+1}-y_\rho; \rho+1<\beta-1) \\ &\qquad \geqq (y_{\beta-1}-y_{\beta-2}) \oplus \bigvee(y_\alpha;\ \alpha<\beta-1) \\ &\qquad = (y_{\beta-1}-y_{\beta-2}) \oplus y_{\beta-2} = y_{\beta-1}, \end{aligned}$$

which proves $P(\beta)$. (In the second to the last step we used the induction hypothesis.)

THEOREM. *An m-orthocomplete orthomodular lattice is m-complete.*

PROOF. By induction. Let $(x_\gamma;\ \gamma\in\Sigma)$ be a family of elements from $\mathcal{L}$ indexed by a set Σ with $\operatorname{card}(\Sigma)\leqq m$, and assume that the join of any Σ'-indexed family exists when $\operatorname{card}(\Sigma')<\operatorname{card}(\Sigma)$. Let σ be the least ordinal corresponding to $\operatorname{card}(\Sigma)$. We can suppose that $\operatorname{card}(\Sigma)$ is infinite so that σ is a limit ordinal, and we can suppose that we have replaced the set Σ by the set $(\alpha;\ \alpha<\sigma)$ so that we are dealing with an ordinal-indexed family $(x_\alpha;\ \alpha<\sigma)$. By the induction assumption $y_\alpha=\bigvee(x_\rho,\ \rho<\alpha)$ exists for every $\alpha<\sigma$. This family $(y_\alpha;\ \alpha<\sigma)$ satisfies the conditions of the lemma, (i) and (ii) being obviously met, and (iii) being a consequence of the following direct computation for β a limit ordinal $<\sigma$: $\bigvee(y_\alpha;\ \alpha<\beta)=\bigvee_{\alpha<\beta}\bigvee(x_\rho;\ \rho<\alpha)=\bigvee(x_\rho;\ \rho<\beta)$ $=y_\beta$.

The orthogonal join $z=\bigoplus(y_{\alpha+1}-y_\alpha;\ \alpha+1<\sigma)$ exists by m-orthocompleteness; this element z is the desired join, $\bigvee(x_\rho;\ \rho<\sigma)$.

First, note that if z is in fact an upper bound of the set $(x_\rho;\ \rho<\sigma)$, then, among all such upper bounds, it is certainly the least. For if $w\geqq x_\rho$ for all $\rho<\sigma$, then $w\geqq\bigvee(x_\rho;\ \rho<\alpha+1)=y_{\alpha+1}\geqq y_{\alpha+1}-y_\alpha$ for all

$\alpha+1<\sigma$ so $w\geqq z$. Hence it is enough to show that $z\geqq x_\beta$ for every $\beta<\sigma$.

If $\beta<\sigma$ then, σ being a limit ordinal, we have $\beta+2<\sigma$, whence

$$\begin{aligned} x_\beta &\leqq \bigvee (x_\rho ; \rho < \beta + 1) = y_{\beta+1} = \bigvee (y_\alpha ; \alpha \leqq \beta + 1) \\ &= \bigvee (y_\alpha ; \alpha < \beta + 2) = \bigoplus (y_{\rho+1} - y_\rho ; \rho + 1 < \beta + 2) \leqq z, \end{aligned}$$

where, in the second-to-the last step, we have used the lemma. That proves the theorem.

Call an orthomodular lattice $\mathfrak{L}$ *orthocomplete* if it is m-orthocomplete for every m (or for $m=\mathrm{card}(\mathfrak{L})$ which is enough).

COROLLARY 1. *An orthocomplete orthomodular lattice is complete.*

An orthomodular lattice $\mathfrak{L}$ satisfies the "m-chain condition" (I am adapting this nomenclature from Sikorski [2, p. 72]) provided that any orthogonal family in $\mathfrak{L}$ has $\leqq m$ nonzero elements.

COROLLARY 2. *An m-orthocomplete orthomodular lattice satisfying the m-chain condition is complete.*

For $m=\aleph_0$, this was proved by Zierler [3, Lemmas 1.8 and 1.9].

COROLLARY 3 (SMITH-TARSKI; SEE [2; §20.1]). *An m-orthocomplete Boolean algebra is m-complete.*

COROLLARY 4 (TARSKI; SEE [2; §20.5]). *An m-orthocomplete Boolean algebra satisfying the m-chain condition is complete.*

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