EXTENSIONS OF FREE GROUPS BY TORSION GROUPS

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Recently R. G. Swan [5] has proven the following theorem with no commutative assumptions whatsoever: If a torsion free group G contains a free subgroup of finite index, then G is free. Of course, if G is abelian, there is a trivial generalization to the case where G/F is assumed to be bounded with F free (that is, $n(G/F)=0$ where $n\neq 0 \in Z$). However, in this note we show that Swan's theorem has an even more striking improvement than the one suggested above for G an abelian group. For example, G/F may be a subgroup of a totally projective p-group of arbitrary length.

In the ensuing discussion all groups will be abelian and p will be an arbitrary but fixed prime. We denote the group of integers by Z and the group of rational numbers by Q. A cotorsion functor S (defined by Nunke in [3]) is one that has a representing sequence $Z \rightarrowtail M \twoheadrightarrow H$ where H is torsion such that, for a group G, $SG = \text{Image}(\text{Hom}(M, G) \rightarrow \text{Hom}(Z, G) \cong G)$ and if $f: G \rightarrow A$ then $S(f) = f | SG$. We further use the notation S_p when H is a p-group (that is, S_p is a p-coprimary functor in the sense of Nunke) and call S_p reduced if H is reduced. Hill and Megibben [2] call a group G a weak S-projective if for any free resolution $F_0 \rightarrowtail F \twoheadrightarrow G$ of G and any group X we have that $\delta_X(S\ \text{Hom}(F_0, X)) = 0$ where

$$\delta_X: \text{Hom}(F_0, X) \rightarrow \text{Ext}(G, X).$$

If S has enough projectives, then a result of Hill and Megibben [2] shows that a p-group G is a weak S-projective if and only if G is a subgroup of an S-projective. Recall that $pG = \{x \in G : x = py,\ y \in G\}$ and that for an ordinal α, $p^\alpha G = p(p^{\alpha-1}G)$ if $\alpha - 1$ is defined and $p^\alpha G = \bigcap_{\beta<\alpha} p^\beta G$ if α is a limit ordinal. The functors p^α comprise the most well-known and studied examples of the functors described above. Hereafter, we let $\mathcal{S}_p$ be the class of all p-groups that are weak S_p-projective for some reduced cotorsion functor S_p. This class contains (in increasing generality) the following classes of p-groups:

(1) Arbitrary direct sums of cyclic p-groups.

(2) Subgroups of arbitrary direct sums of countable reduced p-groups.

(3) Subgroups of totally projective p-groups of arbitrary length.[1]

Before proving the result promised in the introduction, we state

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[1] A p-group G is totally projective if and only if $G/p^\alpha G$ is a p^α-projective for all ordinals α.

a slight generalization of the author's Theorem 2.3 [1]. The proof carries over almost verbatim.

THEOREM. *If $[M_i]_{i\in I}$ is a family of groups of finite torsion free rank such that the torsion free subgroups of each M_i are free, then each torsion free subgroup of $\sum_{i\in I} M_i$ is free.*

We remark here that the above theorem can easily be further generalized to the case where each M_i is assumed only to have countable torsion free rank.

THEOREM. *If G is a torsion free group such that G contains a free subgroup F with $G/F\in \mathcal{S}_p$, then G is free.*

PROOF. Let S_p denote the reduced cotorsion functor for which G/F is a weak S_p-projective and let $Z \rightarrowtail M \twoheadrightarrow H$ be its representing sequence. We remark that $Z\subseteq S_pM$ and that each torsion free subgroup of M is isomorphic to a subgroup of Z. We now form the exact sequence $\sum Z \rightarrowtail \sum M \twoheadrightarrow \sum H$ so that $F\cong \sum Z$. Since S_p commutes with sums, there is an imbedding $\psi:\Sigma Z\hookrightarrow S_p(\sum M)$; hence there is a diagram

$$\begin{tikzcd} F \arrow[r, tail] \arrow[d, "\psi"'] & G \arrow[r] & G/F \\ S_p(\sum M) \arrow[r, tail] & \sum M. & \end{tikzcd}$$

Quoting Hill and Megibben's Theorem 2.2 [2] yields a homomorphism $\phi:G\rightarrow \sum M$ which makes the above diagram commutative. Hence $\operatorname{Ker} \phi\cap F=0$ which implies that ϕ is monic since F is necessarily an essential subgroup of G. An application of the author's theorem (above) shows that G is free.

COROLLARY. *Let S_p be a reduced cotorsion functor with enough projectives. Then each torsion free weak S_p-projective is free.*

PROOF. Let A be a torsion free weak S_p-projective. From Hill and Megibben's Theorems 2.8 and 2.9 [2], we conclude that there is an exact sequence $F\rightarrowtail A\twoheadrightarrow B$ where F is free and B is a reduced weak S_p-projective p-group. Thus the theorem shows that A is free.

Hill and Megibben [2] have shown that the class of weak S_p-projectives is closed under the operations of taking direct sums and of taking subgroups. Hence, one can observe that the assumption on the rank of the groups M_i in our first theorem can be dropped if each M_i is assumed to be S_p-projective for some fixed reduced cotorsion functor S_p.

COROLLARY. *If a torsion free group G contains a free subgroup F such that G/F is contained in a sum of countable reduced p-groups, then G is free.*

Our insistance that only one prime be allowed rests upon the fact that there is an immediate counterexample when infinitely many primes are used. For example, for N an infinite subset of the primes there is a noncyclic group B such that $Z\subseteq B\subseteq Q$ and $B/Z\cong\sum_{p\in N} Z(p)$. However, it is clear that the above results remain true when G/F has a finite number of primary components and each of these components is an appropriate weak projective. Although it remains open as to whether G is necessarily free when G/F is only assumed to be a reduced p-group, we can make some comment on the general situation. A group G is called $\aleph_1$-free if each countable subgroup of G is free and G is called slender (see [4]) if G contains no copy of $Q, \prod_{\aleph} Z$ or the q-adic integers for any prime q.

THEOREM. *If a torsion free group G is an extension of a free group F by a reduced p-group, then G is $\aleph_1$-free and slender.*

PROOF. If C is a countable subgroup of G, then applying the above corollary to the exact sequence $C\cap F\rightarrowtail C\twoheadrightarrow\{C, F\}/F\subseteq G/F$ shows that C is free. Hence G is $\aleph_1$-free. To show that an $\aleph_1$-free group is slender, it suffices to show that $\prod_{\aleph_0} Z$ cannot be isomorphic to a subgroup of G. So suppose that $P\cong\prod_{\aleph_0} Z$ is a subgroup of G. From the exact sequence $F\cap P\rightarrowtail P\twoheadrightarrow\{P, F\}/F\subseteq G/F$ and a result of Nunke [4], we obtain that $\{P, F\}/F$ is bounded which further implies the contradictory fact that P is free.

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