ON SLOW VARIATION

GÉRARD LETAC

A positive function L on the positive real line is said to be "slowly varying at infinity" if, for each $t>0$:

$$\lim_{x\to\infty} \frac{L(xt)}{L(x)} = 1, \qquad t \in (0, \infty).$$

Karamata [3] has proved that if L is continuous, then

$$L(x) = a(x) \exp\left(\int_1^x \frac{\epsilon(y)}{y} dy \right) \tag{1}$$

where $\epsilon(x) \to 0$ and $a(x) \to c \in (0, \infty)$ as $x \to +\infty$. Feller [2, pp. 272–274] gives a new exposition of the theory and a proof of (1), implicitly assuming not the continuity, but the local integrability of L on some half line (A, ∞). But it has been already proved [1], [4][1] that measurability of L is enough.

From (1), it follows that [2, footnote p. 302]:

$$\lim_{x\to+\infty} x^{\alpha} L(x) = \infty, \qquad \lim_{x\to+\infty} x^{-\alpha} L(x) = 0 \quad (\alpha > 0). \tag{2}$$

The aim of this note is to show that L measurable implies that L is locally bounded on some half line (A, ∞) (thus preparing for Feller's exposition) and to give a short proof of (2) which avoids an appeal to (1), by establishing the following theorem:

THEOREM. *If L is slowly varying and measurable, then for every $\alpha > 0$, there exists $X(\alpha)$ and $T(\alpha)$ such that $x > X(\alpha)$ and $t > T(\alpha)$ imply*:

$$t^{-\alpha} \leqq L(xt)/L(x) \leqq t^{\alpha}.$$

PROOF. Let $S_n = \{t > 1 : t^{-\alpha} \leqq L(xt)/L(x) \leqq t^{\alpha} \; \forall x > n\}$.
From slow variation it follows:

$$\bigcup_{n=1}^{\infty} S_n = \{t : t > 1\}.$$

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[1] The author is indebted to the referee for these references.

Since L is measurable, there exists n_0 such that S_{n_0} has a positive Lebesgue measure. Now $S_n S_n \subset S_n$, i.e. S_n is a multiplicative semigroup. Hence the interior of S_{n_0} is not empty; this implies that S_{n_0} contains a half-line $(T(\alpha), \infty)$, and we can take $X(\alpha) = n_0$.

The idea of the proof can be used [5] to get uniform convergence of $L(xt)/L(x)$ on compact subsets of R^+.

Bibliography

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Université de Montréal