METRIC DIMENSION OF COMPLETE METRIC SPACES[1]

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1. Introduction and results. For integers $n \geqq 3$, let (X_n, ρ) be a metric space such that

(i) $X_n \subset (K_n, \rho)$, a compact n-dimensional metric space;

(ii) $X_n = K_n - \bigcup_{i=1}^{\infty} A_i$, where the A_i's are mutually disjoint and closed in K_n; and

(iii) $\mu \dim(X_n, \rho) = [n/2]$ and $\dim X_n = n-1$.

(Here μ dim denotes metric dimension, which is defined in the next section, and dim denotes covering dimension.) K. Sitnikov [8, p. 23] and K. Nagami and J. H. Roberts [6, p. 426] have constructed such spaces.

The result of the present paper is stated in the following theorem.

THEOREM. *For integers $n \geqq 3$, let (X_n, ρ) be a metric space with properties* (i)–(iii) *above. Then there exists a complete metric σ on X_n equivalent to ρ such that*

$$\mu \dim(X_n, \sigma) \leqq [n/2] + 1.$$

K. Nagami and J. H. Roberts posed the following question. Is $\mu \dim(X, d) = \dim X$ for all complete metric spaces (X, d)? In [1, p.166] Richard E. Hodel posed an analogous question. Is $d_2(X, d) = \dim X$ for all complete metric spaces (X, d)? (The metric-dependent dimension function d_2 is defined in the next section.) It is known (see [6, Theorem 4, p. 422]) that $d_2(X, d) \leqq \mu \dim (X, d)$ for all metric spaces (X, d). The present theorem gives a negative answer to these questions, since for $n \geqq 5$,

$$\mu \dim(X_n, \sigma) \leqq [n/2] + 1 < n - 1 = \dim X_n.$$

M. Katětov [4, p. 166] proved that $\dim X \leqq 2 \mu \dim(X, d)$ for all nonempty metric spaces (X, d). In view of this result of Katětov and the present theorem, the following problem is suggested.

PROBLEM. For integers $n \geqq 3$, do there exist complete metric spaces (X_n, d) with $\mu \dim(X_n, d) = [n/2]$ and $\dim X_n = n-1$?

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2. **Definitions.** In this paper three metric-dependent dimension functions are considered:

(i) metric dimension, denoted by μ dim;

(ii) d_2, introduced by K. Nagami and J. H. Roberts in [**5**, p. 602]; and

(iii) d_5, introduced by Richard E. Hodel in [**3**, p. 83].

Metric dimension, d_2, and d_5 are functions from the class of all metric spaces (X, d) into $\{-1, 0, 1, \cdots; \infty\}$. Condensed definitions of these functions restricted to nonempty metric spaces are as follows.

DEFINITION. μ $\dim(X, d)$ is the smallest integer n such that for all $\epsilon>0$ there exists an open cover $\mathfrak{U}(\epsilon)$ of X with (1) order $\mathfrak{U}(\epsilon) \leqq n+1$ and (2) mesh $\mathfrak{U}(\epsilon)<\epsilon$.

DEFINITION. $d_2(X, d)$ is the smallest integer n such that given any $n+1$ pairs $\{C_i, C_i'\}_{i=1}^{n+1}$ of closed sets with $d(C_i, C_i')>0$ for each i, there exist closed sets $\{B_i\}_{i=1}^{n+1}$ such that

(i) B_i separates C_i and C_i' in X for each i and

(ii) $\bigcap_{i=1}^{n+1} B_i=\emptyset$.

DEFINITION. $d_5(X, d)$ is the smallest integer n such that given any countable number of pairs $\{C_i, C_i'\}_{i=1}^{\infty}$ of closed sets with $d(C_i, C_i')\geqq\delta$ for each i for some $\delta>0$, there exist closed sets $\{B_i\}_{i=1}^{\infty}$ such that

(i) B_i separates C_i and C_i' in X for each i and

(ii) order $\{B_i\}_{i=1}^{\infty}\leqq n$.

3. **Proof of the theorem.**

3.1. REDUCING THE PROBLEM. Fix an integer $n\geqq 3$. Let (X_n, ρ) be a metric space with properties (i)–(iii) above. We may assume that every A_i is nonempty. Define

$$f_i(x) = \frac{1}{\rho(x, A_i)}, \qquad (x \in K_n - A_i,\ i \geqq 1);$$

$$\alpha_i(x, y) = 2^{-i}\cdot\frac{|f_i(x) - f_i(y)|}{1 + |f_i(x) - f_i(y)|}, \quad (x, y \in K_n - A_i,\ i \geqq 1);$$

$$\sigma(x, y) = \rho(x, y) + \sum_{i=1}^{\infty} \alpha_i(x, y), \qquad (x, y \in X_n).$$

It is known (see [2, Theorem 2-76, p. 85]) that σ is a complete metric on X_n equivalent to ρ.

We shall prove that μ $\dim(X_n, \sigma)\leqq[n/2]+1$. It is proved in [**3**, p. 85] that $d_5(X, d)=\mu$ $\dim(X, d)$ for all separable metric spaces (X, d). Now X_n is separable, so it suffices to prove that $d_5(X_n, \sigma) \leqq [n/2]+1$. Let $\{C_i, C_i'\}_{i=1}^{\infty}$ be a countable number of pairs of closed sets in X_n with $\sigma(C_i, C_i')\geqq\epsilon$ for each i for some $\epsilon>0$. We want to show that there exist closed sets $\{B_i\}_{i=1}^{\infty}$ in X_n such that

(i) B_i separates C_i and C_i' in X_n for each i and

(ii) order $\{B_i\}_{i=1}^{\infty}\leqq[n/2]+1$.

Since $\sum_{i=1}^{\infty}\alpha_i$ converges uniformly in X_n, there exists an integer $N>1$ such that $\sum_{i=N+1}^{\infty}\alpha_i(x, y)<\epsilon/2$ for all $x, y\in X_n$. Define

$$\sigma^N(x, y) = \rho(x, y) + \sum_{i=1}^{N}\alpha_i(x, y), \qquad (x, y \in X_n)\cdot$$

$$A = \bigcup_{i=1}^{N} A_i.$$

Then clearly σ^N is a metric on X_n equivalent to ρ. Also, since $\sigma(C_i, C_i')\geqq\epsilon$ for all i, it follows that

$$\sigma^N(C_i, C_i') \geqq \frac{\epsilon}{2} \quad \text{for all } i. \tag{1}$$

3.2. Definitions. Define

$$\delta = \min\{\rho(A_i, A_j): i, j \in \{1, 2, \cdots, N\}, i \neq j\},$$

$$\gamma = \min\left\{\frac{\delta}{4}, \frac{\epsilon}{6}, \frac{\epsilon\delta^2}{24(N-1)}\right\}.$$

3.3. Assertion 1. For all numbers a such that $0<a\leqq\delta/4$, there exists an $\epsilon(a)>0$ such that $\rho(C_i, C_i')\geqq\gamma$ in $S(\epsilon(a))$ $(\equiv\{x\in K_n: a-\epsilon(a)<\rho(x, A)<a+\epsilon(a)\})$ for $i\geqq 1$.

Proof. Fix a such that $0<a\leqq\delta/4$. Choose $\epsilon(a)>0$ such that $\epsilon(a)<\min\{a/2, \epsilon a^2/48\}$. Suppose there exists an integer $i\geqq 1$ such that $\rho(C_i, C_i')<\gamma$ in $S(\epsilon(a))$. Then there exist points $x\in C_i$ and $y\in C_i'$ such that $\{x, y\}\subset S(\epsilon(a))$ and $\rho(x, y)<\gamma$. From the definition of γ and the choice of $\epsilon(a)$, it follows that $\rho(x, y)<\delta/4$, $\rho(x, A)<3\delta/8$, and $\rho(y, A)<3\delta/8$. Therefore by the definition of δ, there exists an integer $k\in\{1, 2, \cdots, N\}$ such that $\rho(x, A_k)<3\delta/8$ and $\rho(y, A_k)<3\delta/8$. Thus for $i\in\{1, 2, \cdots, N\}$ and $i\neq k$, $\rho(x, A_i)>\delta/2$ and $\rho(y, A_i)>\delta/2$. It follows that $a-\epsilon(a)<\rho(x, A_k)<a+\epsilon(a)$ and $a-\epsilon(a)<\rho(y, A_k)<a+\epsilon(a)$. Hence $|\rho(x, A_k)-\rho(y, A_k)|<2\epsilon(a)$. Finally, $\rho(x, A_k)>a/2$ and $\rho(y, A_k)>a/2$. From the definitions of σ^N and γ and the inequalities above, it follows that

$$\begin{aligned}
\sigma^N(x, y) &\leqq \rho(x, y) + \sum_{i=1}^{N} |f_i(x) - f_i(y)| \\
&\leqq \rho(x, y) + \sum_{i=1}^{N} \frac{|\rho(x, A_i) - \rho(y, A_i)|}{\rho(x, A_i)\cdot\rho(y, A_i)} \\
&\leqq \rho(x, y) + \sum_{\substack{i=1 \\ i\neq k}}^{N} \frac{\rho(x, y)}{\rho(x, A_i)\cdot\rho(y, A_i)} + \frac{|\rho(x, A_k) - \rho(y, A_k)|}{\rho(x, A_k)\cdot\rho(y, A_k)} \\
&< \gamma + \frac{(N-1)\gamma}{\delta^2/4} + \frac{2\epsilon(a)}{a^2/4} \\
&< \epsilon/6 + \epsilon/6 + \epsilon/6 = \epsilon/2,
\end{aligned}$$

contradicting (1).

3.4. Construction of C_{ij}, C'_{ij}. Now (i) $\{S(\epsilon(a)):0<a\leqq\delta/4\}$ is a collection of open sets in K_n covering $\{x\in K_n:0<\rho(x, A)\leqq\delta/4\}$ and (ii) $\{x\in K_n:\delta/(4\cdot 2^j)\leqq\rho(x, A)\leqq\delta/(4\cdot 2^{j-1})\}$ is compact for $j\geqq 1$. Using (i) and (ii), it is easy to prove that there exist a sequence $\{a_j\}_{j=1}^{\infty}$ of positive numbers $\leqq\delta/4$ such that

(a) $\bigcup_{j=1}^{\infty} S(\epsilon(a_j))$ covers $\{x\in K_n:0<\rho(x, A)\leqq\delta/4\}$ and

(b) the sequence $\{a_j\}_{j=1}^{\infty}$ converges to 0.

We can choose a sequence $\{\delta_j\}_{j=1}^{\infty}$ of distinct positive numbers such that $\delta_1=\delta/4$, $\{\delta_j\}_{j=1}^{\infty}$ is a strictly decreasing sequence converging to 0, and for each $j\geqq 2$ there exists an integer $k\geqq 1$ such that

$$a_k - \epsilon(a_k) < \delta_{j+1} < \delta_{j-1} < a_k + \epsilon(a_k). \tag{2}$$

Now we define distinct positive numbers $\{\delta_{ij}\}_{i,j=1}^{\infty}$ as follows. Fix $j\geqq 1$. Define $\delta_{1j}=\delta_j$. For $i>1$ choose the δ_{ij}'s to be distinct numbers strictly between δ_j and δ_{j+1}.

Now define

$$\begin{aligned}
E_{i1} &= \{x\in X_n:\ \rho(x, A)\geqq\delta_{i1}\}, \qquad (i\geqq 1); \\
E_{ij} &= \{x\in X_n:\ \delta_{ij}\leqq\rho(x, A)\leqq\delta_{i,j-1}\}, \qquad (i\geqq 1, j>1); \\
C_{ij} &= C_i\cap E_{ij}, \qquad C'_{ij} = C'_i\cap E_{ij}, \qquad (i, j\geqq 1).
\end{aligned}$$

3.5. Assertion 2. There exists a $\tau>0$ such that $\rho(C_{ij}, C'_{ij})\geqq\tau$ for $i, j\geqq 1$.

Proof. Define $\tau=\min\{\gamma, \epsilon\delta_2^2/4N\}$.

Case 1. $j=1$. Suppose there exists an integer $i\geqq 1$ such that

$\rho(C_{i1}, C'_{i1}) < \tau$. Let $x \in C_{i1}$ and $y \in C'_{i1}$ be such that $\rho(x, y) < \tau$. Note that $\rho(x, A) > \delta_2$ and $\rho(y, A) > \delta_2$, since $\{x, y\} \subset E_{i1}$. Hence

$$\begin{aligned} \sigma^N(x, y) &\leqq \rho(x, y) + \sum_{i=1}^{N} \frac{|\rho(x, A_i) - \rho(y, A_i)|}{\rho(x, A_i)\cdot\rho(y, A_i)} \\ &\leqq \rho(x, y) + \sum_{i=1}^{N} \frac{\rho(x, y)}{\rho(x, A_i)\cdot\rho(y, A_i)} \\ &< \tau + N\tau/\delta_2^2 \\ &< \epsilon/4 + \epsilon/4 = \epsilon/2, \end{aligned}$$

a contradiction to (1).

Case 2. $j > 1$. Fix $i \geqq 1$ and $j > 1$. Now by the definition of E_{ij} and by (2),

$$\begin{aligned} E_{ij} &\subset \{x \in X_n: \delta_{j+1} \leqq \rho(x, A) \leqq \delta_{j-1}\} \\ &\subset S(\epsilon(a)) \end{aligned}$$

for some a such that $0 < a \leqq \delta/4$. Therefore by the definitions of C_{ij} and C'_{ij} and Assertion 1, $\rho(C_{ij}, C'_{ij}) \geqq \gamma \geqq \tau$.

3.6. Lemma [7]. *Let X be a topological space, let C and C' be disjoint closed sets in X, and let $\{D_j\}_{j=0}^{\infty}$ be an open cover of X such that $D_0 = \varnothing$ and $\overline{D}_j \subset D_{j+1}$ for all $j \geqq 1$. Suppose there exist closed sets $\{B_j\}_{j=1}^{\infty}$ in X such that $B_j \subset \overline{D}_j - D_{j-1}$ for $j \geqq 1$ and B_j separates $C \cap (\overline{D}_j - D_{j-1})$ and $C' \cap (\overline{D}_j - D_{j-1})$ in $\overline{D}_j - D_{j-1}$ for $j \geqq 1$. Then there exists a closed set B in X such that B separates C and C' in X and $B \subset \bigcup_{j=1}^{\infty} (B_j \cup (\overline{D}_j - D_j))$.*

3.7. Conclusion of the proof of the theorem. By Assertion 2 and the equality $d_5(X_n, \rho) = [n/2]$, there exist closed sets $\{B'_{ij}\}_{i,j=1}^{\infty}$ in X_n such that B'_{ij} separates C_{ij} and C'_{ij} in X_n for $i, j \geqq 1$ and order $\{B'_{ij}\}_{i,j=1}^{\infty} \leqq [n/2]$. For $i \geqq 1$ define $D_{i0} = \varnothing$. For $i, j \geqq 1$ define $D_{ij} = \{x \in X_n: \rho(x, A) > \delta_{ij}\}$ and $B_{ij} = B'_{ij} \cap (\overline{D}_{ij} - D_{i,j-1})$, where for every i and j the closure of D_{ij} is taken with respect to X_n. Then clearly B_{ij} separates C_i and C'_i in $\overline{D}_{ij} - D_{i,j-1}$ for $i, j \geqq 1$ and

$$\text{order}\{B_{ij}\}_{i,j=1}^{\infty} \leqq [n/2]. \tag{3}$$

Now fix $i \geqq 1$. Clearly X_n, C_i, C'_i, $\{D_{ij}\}_{j=0}^{\infty}$, and $\{B_{ij}\}_{j=1}^{\infty}$ satisfy the conditions of the lemma. Therefore there exists a closed set B_i in X_n such that B_i separates C_i and C'_i in X_n and

$$B_i \subset \bigcup_{j=1}^{\infty} (B_{ij} \cup (\overline{D}_{ij} - D_{ij})).$$

But for $j \geqq 1$,

$$\bar{D}_{ij} - D_{ij} \subset \{x \in X_n: \rho(x, A) = \delta_{ij}\}.$$

Hence

$$B_i \subset \bigcup_{j=1}^{\infty} (B_{ij} \cup \{x \in X_n: \rho(x, A) = \delta_{ij}\}).$$

Therefore, by (3) and the fact that the δ_{ij}'s are distinct for $i,j \geqq 1$, we have that order $\{B_i\}_{i=1}^{\infty} \leqq [n/2]+1$, and the proof is complete.

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