A FIVE SPHERE DECOMPOSITION OF E^{2n-1}

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I. **Introduction.** R. H. Bing and M. L. Curtis have exhibited a decomposition of Euclidean 3-dimensional space E^3 into twelve mutually disjoint circles and points not on the circles such that the associated decomposition space can not be embedded in E^4 [1]. Their method consists in showing that the space contains a certain 2-dimensional polyhedron that Flores has proved to be impossible to embed in E^4 [2]. The construction of Bing and Curtis was later modified by R. H. Rosen, who, by improving the result of Flores, also exhibited a decomposition of E^3 that can not be embedded in E^4, and in which he used only six circles instead of twelve [4]. In the opposite direction, R. P. Goblirsch showed that every decomposition using only three circles as nondegenerate elements can be embedded in E^4 [3]. Thus, for the numbers four and five the question remained open. Rosen conjectured in [4] that one could build an example by using five circles in E^3 such that each circle links exactly two others. In this paper we show this conjecture to be correct. Moreover, our argument begins in a lower dimension: We construct an analogous decomposition of S^1 with five nontrivial elements such that the associated decomposition space can not be embedded in S^2. The example conjectured by Rosen then becomes the second step in an induction argument. Thus we show that for each integer n, $n \geqq 1$, there exists a decomposition of S^{2n-1} with nondegenerate elements consisting of five $(n-1)$-spheres such that the associated decomposition space can not be embedded in S^{2n}. This inductive viewpoint was inspired by a paper of Joseph Zaks [5], in which decompositions of E^{2n-1} with finitely many nondegenerate elements were constructed for all $n \geqq 1$.

II. **Embedding an n-complex in S^{2n-1}.** Let N^1 denote the 1-skeleton of a 4-simplex with vertices a_1, b_1, c_1, d_1, and e_1. Let N^2 denote the join $V(N^1, \{a_2, b_2, c_2\})$ of N^1 with the three point space $\{a_2, b_2, c_2\}$. Proceeding inductively, N^n is defined as $V(N^{n-1}, \{a_n, b_n, c_n\})$. It is shown in [2] and [4] that N^n can not be embedded in E^{2n}. We name five n-simplices of N^n:

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$$D_1 = a_1c_1c_2c_3 \cdots c_{n-1}c_n$$
$$D_2 = a_1d_1c_2c_3 \cdots c_{n-1}c_n$$
$$D_3 = b_1d_1a_2a_3 \cdots a_{n-1}a_n$$
$$D_4 = b_1e_1a_2a_3 \cdots a_{n-1}a_n$$
$$D_5 = c_1e_1b_2b_3 \cdots b_{n-1}b_n$$

Setting $N^n_- = N^n - \sum_1^5 \text{Int } D_i$, we find that N^n_- embeds in S^{2n}. In fact, it embeds in S^{2n-1}! Rather than prove this fact, which would require cumbersome notation, we establish a weaker result, which suffices for our purposes. We call two points of a geometric complex *distant* if they lie in disjoint, closed simplexes of the complex.

LEMMA. *For $n \geqq 1$, there exists a map $f_n: N^n_- \to S^{2n-1}$ such that no two distant points of N^n_- have the same image.*

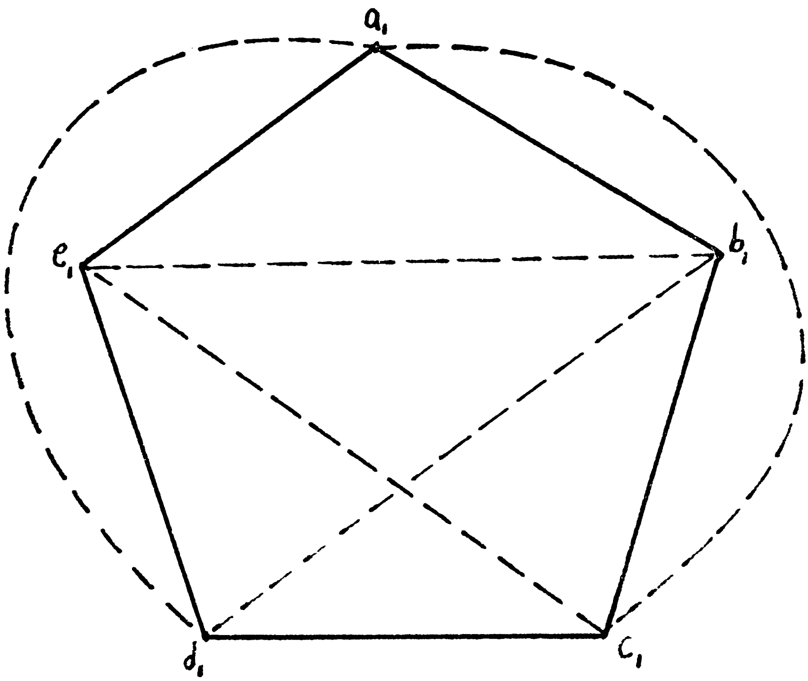

FIGURE 1

PROOF. An induction argument begins with the fact that N^1_- is homeomorphic to S^1 as is shown in Figure 1; call such a homeomorphism f_1. For $n=2$, the reader is advised first to familiarize himself

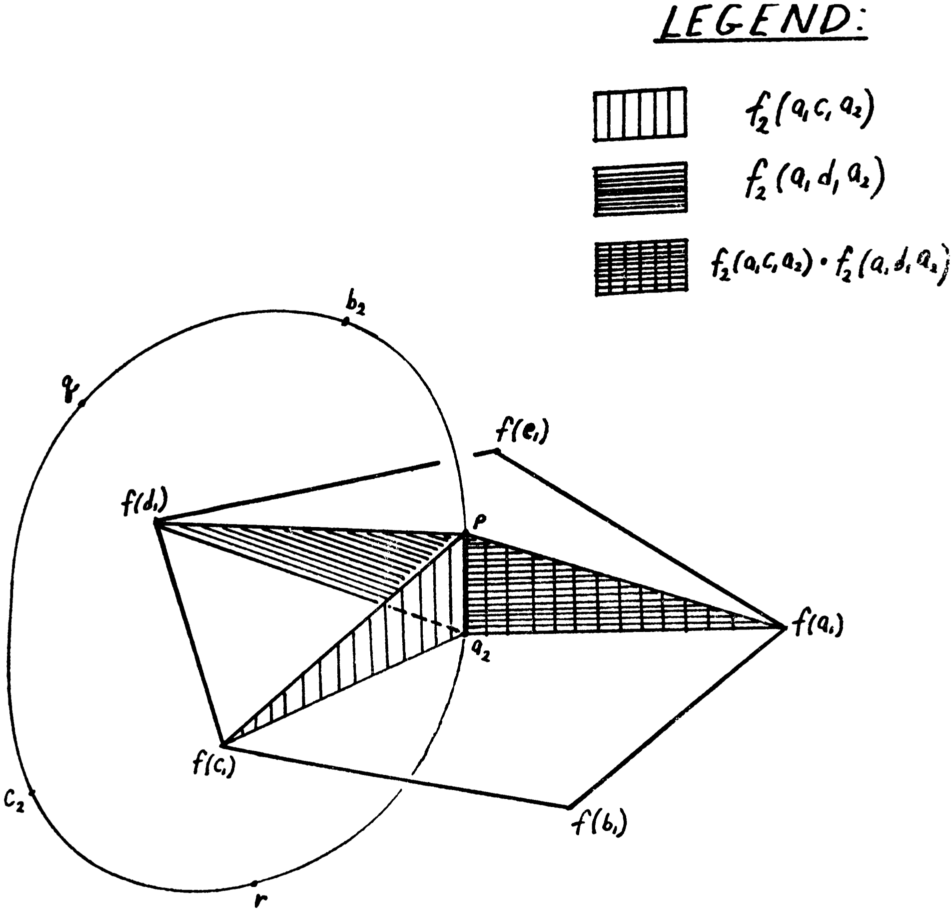

FIGURE 2

with the visualizations given in [**1**]. In fact, for $n=2$, Bing and Curtis construct geometrically just what we will do notationally, except that their complex "lacks" three 2-cells instead of the five 2-cells that N^2_- "lacks." We regard S^3 as the join $V(S^1, S^1)$, with f_1 viewed as an embedding of N^1_- into the first factor of $V(S^1, S^1)$, and with $\{a_2, b_2, c_2\}$ viewed as a subset of the second factor. Then $V(f_1(N^1_-), \{a_2, b_2, c_2\})$ is a subset of $V(S^1, S^1)$ in a natural way; this provides us with an embedding f_2 of all but ten 2-simplices of N^2_- into S^3. We select points p, q, and r in the second factor of $V(S^1, S^1)$ so that this factor is composed of the six arcs a_2p, pb_2, b_2q, qc_2, c_2r, ra_2. We define $f_2(a_1c_1)$ as $V(f_1(\text{Bd } a_1c_1), p)$, $f_2(a_1c_1a_2)$ as $V(f_1(\text{Bd } a_1c_1), a_2p)$ as illustrated in

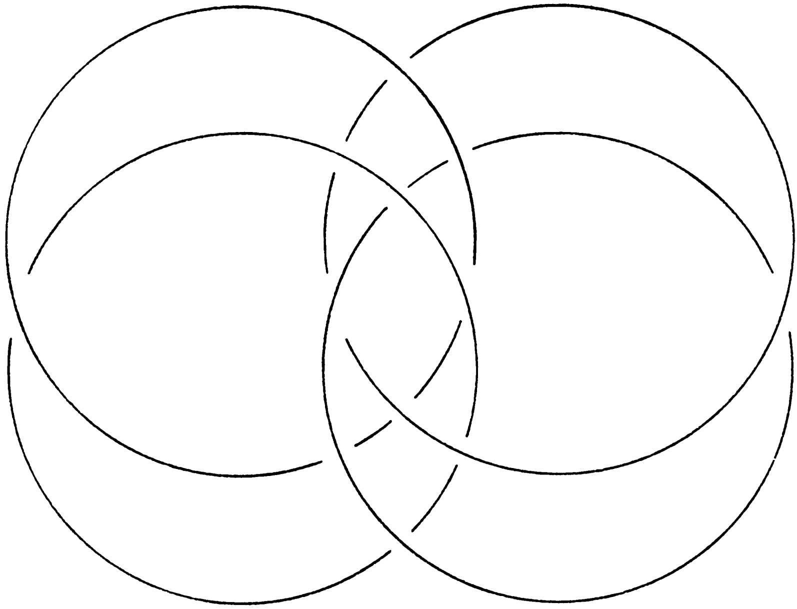

FIGURE 3

Figure 2, and $f_2(a_1c_1b_2)$ as $V(f_1(\text{Bd } a_1c_1), pb_2)$. Next, we define $f_2(a_1d_1)$ as $V(f_1(\text{Bd } a_1d_1), p)$, $f_2(a_1d_1a_2)$ as $V(f_1(\text{Bd } a_1d_1), a_2p)$ as illustrated in Figure 2, and $f_2(a_1d_1b_2)$ as $V(f_1(\text{Bd } a_1d_1), pb_2)$. Similarly, we define $f_2(b_1d_1)$ as $V(f_1(\text{Bd } b_1d_1), q)$ and insert $f_2(b_1d_1b_2)$ and $f_2(b_1d_1c_2)$, $f_2(b_1e_1)$ as $V(f_1(\text{Bd } b_1e_1), q)$ and insert $f_2(b_1e_1b_2)$ and $f_2(b_1e_1c_2)$. Lastly, we define $f_2(c_1e_1)$ as $V(f_1(\text{Bd } c_1e_1), r)$, then insert $f_2(c_1e_1c_2)$ and $f_2(c_1e_1a_2)$. Thus f_2 has been defined, and one may verify that it satisfies the lemma; in fact, a small adjustment would make f_2 an embedding.

For $n=3$, we let f_2 map into the first factor of $V(S^3, S^1)$, and a_3, p', b_3, q', c_3, r' be consecutive points in the second factor. Then $f_3(a_1c_1c_2)$ is defined as $V(f_2(\text{Bd } a_1c_1c_2), p')$; then $f_3(a_1c_1c_2a_3)$ and $f_3(a_1c_1c_2b_3)$ are inserted as before. The continuation is just a notational exercise.

III. Insertion of five annuli.

THEOREM. *For each integer n, $n \geqq 1$, there exists a decomposition of S^{2n-1} with nondegenerate elements consisting of five $(n-1)$-spheres such that the associated decomposition space can not be embedded in S^{2n}.*

PROOF. Let A' denote the subarc of S^1 with interior point $f_1(a_1)$

and end points $f_1(e_1)+f_1(b_1)$; similarly B' has interior point $f_1(b_1)$ and end points $f_1(a_1)+f_1(c_1)$; analogously we define C', D', and E'. We set $A=V(A', S^{2n-3})\subset S^{2n-1}$, and similarly for B, C, D, and E. The map $f_n: N^n_- \to S^{2n-1}$ can be extended to N^n so that $f_n(\text{Int } D_1)\subset \text{Int } B$, $f_n(\text{Int } D_2)\subset \text{Int } E$, $f_n(\text{Int } D_3)\subset \text{Int } C$, $f_n(\text{Int } D_4)\subset \text{Int } A$, and $f_n(\text{Int } D_5)\subset D$, with $f_n/\text{Int } D_i$ an embedding for all i. We discard an open disk $\mathcal{O}_i$ from $f_n(\text{Int } D_i)$, leaving an annulus U_i with boundary consisting of $\alpha_i = \text{Bd } f_n(D_i)$ plus another n-sphere which we call β_i. By choosing $\mathcal{O}_i$ sufficiently large, we may ensure that

$$U_1\cdot U_3 = U_1\cdot U_4 = U_2\cdot U_4 = U_2\cdot U_5 = U_3\cdot U_5 = \varnothing,$$

as the corresponding α_i's are disjoint. In fact, for all other pairs $U_i\cdot U_j$ with $i\neq j$, this intersection will be precisely $\alpha_i\cdot\alpha_j$. For example, to see that $U_1\cdot U_2=\alpha_1\cdot\alpha_2$, observe that $U_1-\alpha_1\subset\text{Int } B$, $U_2-\alpha_2 \subset \text{Int } E$, and $\text{Int } B\cdot\text{Int } E=\varnothing$.

We wish to show that the decomposition of S^{2n-1} with nondegenerate elements $\beta_1, \beta_2, \cdots, \beta_5$ does not embed in S^{2n}. We show that this would imply a map of N^n into S^{2n} such that no two distant points of N^n have the same image, contradicting [4]. All that needs to be checked is how the annuli $U_i-\alpha_i$ intersect N^n_- in S^{2n-1}. We already know that they do not intersect each other. Furthermore, it is easy to require that $U_i-\alpha_i$ intersects a simplex Δ of N^n_- only if they share a common vertex, by increasing the size of $\mathcal{O}_i$ if necessary. It remains to show that if $\beta_1\cdot\Delta_1\neq\varnothing$ and $\beta_i\cdot\Delta_2\neq\varnothing$, then Δ_1 and Δ_2 have a common vertex. For notational convenience, assume that $i=1$, so $\beta_1\subset\text{Int } B$. By general position, we may assume that Δ_1 and Δ_2 are both n-simplices on N^n_-. But any two n-simplices in Int B have b_1 as a common vertex.

IV. **Questions.** Let us first observe that our result is the best possible for $n=1$; any decomposition of S^1 with four (or less) nondegenerate elements can be embedded in S^2 without great difficulty. For $n\geqq 2$, however, unsolved problems abound. For example, by using methods of Goblirsch [**3**], one can embed all four circle decompositions of S^3 in S^4 with one exception, illustrated in Figure 3. Can this example also be embedded in S^4? Note that care must be taken in this example that the four circles do not lie on a common torus in S^3; that is, these four circles do not all link each other in the most natural way. Indeed, if they did, the technique of [**3**] would give an embedding.

If we do not require circles but merely simple closed curves, then

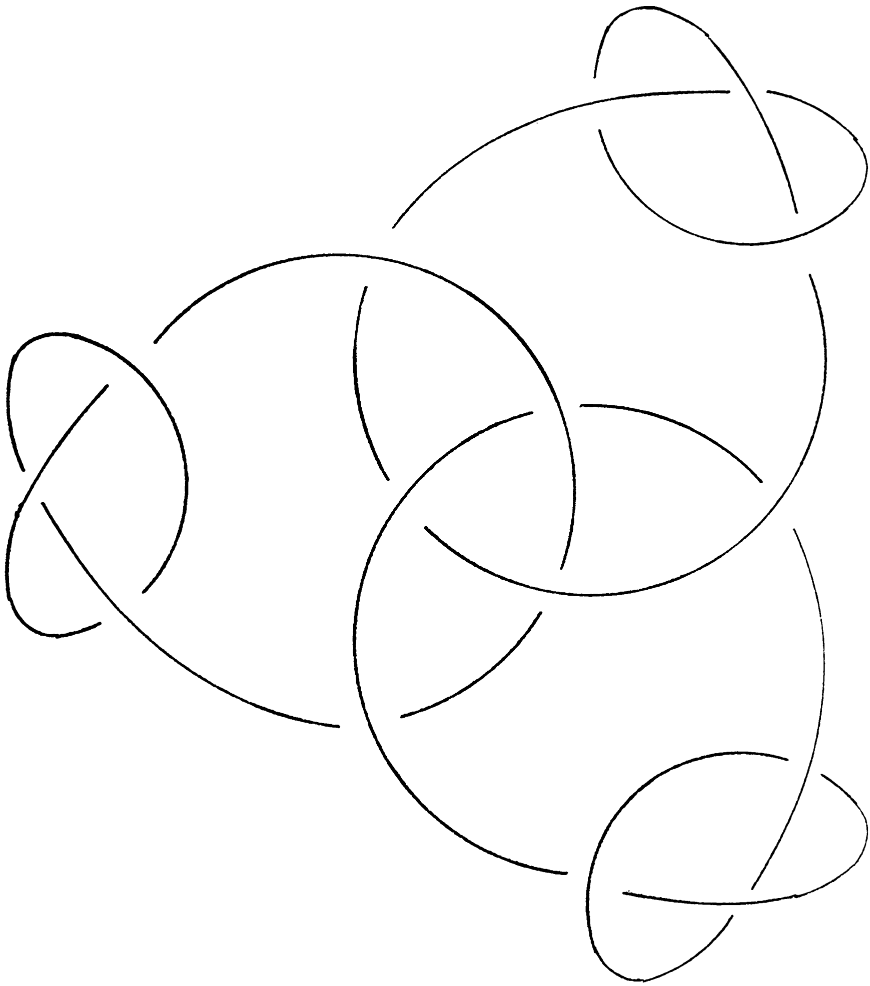

FIGURE 4

Figure 4 gives a decomposition of S^3 with only three nondegenerate sets. Can this example be embedded in S^4? Note that Goblirsch's technique can not be applied to this example. Indeed, this question is unsolved if we do not require simple closed curves, but merely continuua.

If K is an n-complex which locally embeds in S^{2n-1}, does K embed in S^{2n}?

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