OSCILLATION CRITERIA FOR NONLINEAR MATRIX DIFFERENTIAL INEQUALITIES

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ABSTRACT. Oscillation criteria are established for nonlinear matrix differential equations of the form $[A(x)V']' + B(x, V, V')V = 0$ and associated differential inequalities. The hypothesis used recently by E. C. Tomastik, that A and B are positive definite, is weakened to the following: A is positive semidefinite.

Oscillation criteria for the matrix differential equation

$$LV = [A(x)V']' + B(x, V, V')V = 0, \tag{1}$$

and more generally for the inequality $V^T L V \leqq 0$ (as a form), will be derived by a technique different from that given recently by Tomastik [**3**]. It will be assumed that A, B, and V are $m \times m$ matrix functions, $A(x)$ is symmetric, positive semidefinite, and continuous on an interval $[a, \infty)$, and $B(x, V, V')$ is symmetric and continuous for x on $[a, \infty)$ and for all values of the entries of V and V'. Although Tomastik requires $B(x, V, V')$ to be positive definite on $a \leqq x < \infty$ for every matrix V with det $V \neq 0$, we require only that $B(x, V, V')$ satisfies condition (2) below. As already noted, we have weakened the positive definiteness of $A(x)$ to positive semidefiniteness. The technique used here has the advantage that it can be adapted [**2**], [**1**, Chapter 5] to partial differential inequalities of elliptic or parabolic type. As in [**3**] it is assumed that every solution of (1) can be continued to $x = \infty$.

THEOREM 1. *The inequality $V^T L V \leqq 0$ is oscillatory if $A(x)$ is bounded above and there exists a diagonal element B_{ii} of B such that*

$$\int_a^\infty B_{ii}[x, V(x), V'(x)]dx = +\infty \tag{2}$$

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for every differentiable matrix $V(x)$ *with* $\det V(x) \neq 0$ *for all sufficiently large* x.

Proof. Suppose to the contrary that $V^TLV \leqq 0$ is not oscillatory, i.e. [3] there exists a number $b \geqq a$ and a prepared matrix $V(x)$ satisfying $V^TLV \leqq 0$ such that $\det V(x) \neq 0$ in (b, ∞). Then a unique solution $w(x)$ of $u(x) = V(x)w(x)$ exists in (b, ∞) for any m-vector $u(x)$. The following identity is easily verified by differentiation for any piecewise C^1 vector function u:

$$\begin{aligned}&(Vw')^TAVw' + [(Vw)^TAV'w]' \\ &\quad = u^{T'}Au' - u^TBu + u^TLVw + w^T(V^TAV' - V^{T'}AV)w'.\end{aligned}$$

The last term is identically zero since V is prepared [3]. Since $V^TLV \leqq 0$, it follows that

$$F[u, V] \equiv \int_b^c [u^{T'}A(x)u' - u^TB(x, V, V')u]dx \geqq 0 \tag{3}$$

if u is any piecewise C^1 vector function satisfying $u(b) = u(c) = 0$ $(b < c)$. In particular, choose u to be u_i, where

$$\begin{aligned} u_i(x) &= 0 && \text{if } a \leqq x \leqq b \\ &= e_i(x - b) && \text{if } b < x \leqq b + 1 \\ &= e_i && \text{if } b + 1 < x \leqq c - 1 \\ &= e_i(c - x) && \text{if } c - 1 < x \leqq c \end{aligned}$$

$(c > b + 2)$ and e_i is the unit vector with 1 in the ith position and 0 elsewhere. Then

$$\begin{aligned} F(u_i, V) \leqq 2\alpha - \frac{\beta}{3} &- \int_{b+1}^{c-1} B_{ii}(x, V, V')dx \\ &- \int_{c-1}^{c} B_{ii}(x, V, V')(c - x)^2 dx \end{aligned}$$

where α is an upper bound for $A(x)$ on $[b, \infty)$ and β is a lower bound for $B(x, V(x), V'(x))$ on $[b, b+1]$. In view of the hypothesis (2), there exists a number c_0 such that

$$F(u_i, V) \leqq -\int_{c-1}^{c} B_{ii}(x, V, V')(c - x)^2 dx$$

for all $c \geqq c_0$. Define

$$g(w, x) = \int_w^x B_{ii}(t, V(t), V'(t))dt, \qquad w \leqq x.$$

We assert there exists a number $c \geqq c_0$ such that $g(c-1, x) > 0$ for all $x > c-1$. In fact, $g(c_0-1, x)$ has a largest zero $x = c-1$ by (2) $(c \geqq c_0)$ and $g(c_0-1, x) = g(c-1, x) > 0$ for $x > c-1$. An easy integration by parts shows that

$$\int_{c-1}^{c} B_{ii}(x, V, V')(c - x)^2 dx = 2\int_{c-1}^{c} (c - x)g(c - 1, x)dx > 0.$$

Hence $F[u_i, V] < 0$, contradicting (3).

Theorem 1 remains true if (2) is replaced by the obviously stronger condition

$$\int_a^\infty \operatorname{tr} B[x, V(x), V'(x)]dx = + \infty, \tag{4}$$

where tr B denotes the trace of B. In the case that B is positive definite, Theorem 1 also remains true if (2) is replaced by the equivalent condition

$$\int_a^\infty \lambda[x, V(x), V'(x)]dx = + \infty \tag{5}$$

where λ denotes the largest eigenvalue of B. In general, it is clear that (4) can be replaced by (5) provided also $\int_a^\infty \lambda_i(x)dx$ is finite for every negative eigenvalue λ_i of $B[x, V(x), V'(x)]$.

If A is the identity matrix and B is positive definite, the hypothesis (5) is needed only for matrices $V(x)$ such that the smallest eigenvalue of $V^T(x)V(x)$ is bounded away from zero for large x, as shown by Tomastik [**3**].

COROLLARY. *The equation $LV = 0$ is oscillatory if A and B are positive definite, A is bounded above, and* (5) *holds for every differentiable matrix V with* det $V(x) \neq 0$ *for all sufficiently large x.*

This is an obvious special case of Theorem 1; it is also a specialization of Tomastik's Theorem 3 to the case that A is bounded above.

The following generalization of Theorem 1 can be proved in the same way.

THEOREM 2. *$V^T L V \leqq 0$ is oscillatory if for arbitrary $b \geqq a$ there exists a number c $(c > b)$, an integer i, and a piecewise C^1 function ϕ on $[b, c]$ such that $\phi(b) = \phi(c) = 0$ and*

$$\int_b^c \{\phi'^2(x)A_{ii}(x) - \phi^2(x)B_{ii}[x, V(x), V'(x)]\}\,dx < 0$$

for every differentiable matrix $V(x)$ *with* det $V(x) \neq 0$ *on* $[b, \infty)$.

References

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