

SHORTER NOTES

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CUP PRODUCT IN PROJECTIVE SPACES

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ABSTRACT. Cup product in projective spaces is computed by an elementary method.

In two recent algebraic topology texts, [1], [2], the additive cohomology structures of projective spaces are obtained by elementary methods, while cup product is computed by techniques such as the Gysin sequence and the Poincaré duality theorem. We present a computation using basic properties of cup product only.

Think of CP^n as the space of all nonzero complex-coefficient polynomials of the form $P = \alpha_0 + \alpha_1 z + \cdots + \alpha_n z^n$ under identifications $P \sim \lambda P$ for any nonzero complex number λ . Let $M = CP^1 \times \cdots \times CP^1$ (n factors). The multiplication of polynomials defines a map $h: M \rightarrow CP^n$. If D_1, \dots, D_n are mutually disjoint discs in CP^1 , and $D = D_1 \times \cdots \times D_n \subset M$, then by the fundamental theorem of algebra, h is a homeomorphism of D onto $h(D)$, and $K = h^{-1}(h(D))$ is the disjoint union of all $D_\pi = D_{\pi(1)} \times \cdots \times D_{\pi(n)}$, with π ranging through all permutations of $\{1, 2, \dots, n\}$. Note that, if each $(D_\pi, \partial D_\pi)$ is oriented coherently with M , then $\pi: (D, \partial D) \rightarrow (D_\pi, \partial D_\pi)$ preserves orientation.

Using the direct sum decomposition

$$H_{2n}(M, M - K) = \bigoplus_{\pi} H_{2n}(M, M - D_{\pi}),$$

and standard excision arguments, it is not hard to see that $h_*: H_{2n}(M) \rightarrow H_{2n}(CP^n)$, which is the following composite homomorphism

$$H_{2n}(M) \rightarrow H_{2n}(M, M - K) \xrightarrow{h_*} H_{2n}(CP^n, CP^n - h(D)) \xleftarrow{\approx} H_{2n}(CP^n),$$

is multiplication by $\pm n!$

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Let ω denote the generator of $H^2(CP^n)$. Then

$$h^*(\omega) = \sum_{i=1}^n 1 \times \cdots \times \underset{(i\text{th})}{\omega} \times \cdots \times 1 \in H^2(M);$$

$$h^*(\omega^n) = \left(\sum_{i=1}^n 1 \times \cdots \times \omega \times \cdots \times 1 \right)^n = n!(\omega \times \cdots \times \omega).$$

It follows that ω^n generates $H^{2n}(CP^n)$. Consequently $H^*(CP^n) = Z[\omega]/\omega^{n+1}$ as ring.

The ring structure for $H^*(HP^n)$ follows from naturality via the Hopf fibration $CP^{2n+1} \rightarrow HP^n$. By the same technique, it also follows that the 2-dim generator of $H^*(RP^\infty, Z_2)$ generates a polynomial subring. Finally, it is easy to show [2, p. 151, (24.17)] that if $x \in H^1(RP^\infty, Z_2)$ is the 1-dim generator, then $x^2 \neq 0$ in RP^2 . Consequently $H^*(RP^\infty, Z_2) = Z_2[x]$ as ring.

REFERENCES

1. E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966. MR 35 #1007.
2. M. J. Greenberg, *Lectures on algebraic topology*. Benjamin, New York, 1967. MR 35 #6137.

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