A THEOREM OF ASANO AND MICHLER

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There is a theorem due to Asano [**1**, Satz 2.12] which is easily seen to be equivalent to the following

THEOREM. *Let R be a right and left noetherian prime ring with* 1 *such that*

(i) *the fractional two-sided R-ideals form a group,*

(ii) *each essential one-sided ideal contains a nonzero two-sided ideal, and*

(iii) *for each nonzero prime ideal P of R, R/P is simple artinian. Then R is right and left hereditary.*

In a recent paper [**3**], Michler showed that (iii) is a consequence of the other assumptions and therefore can be omitted. Asano's proof of the theorem is quite long and the purpose of this note is to present a new short proof which forms a natural sequel to Michler's work.

First we recall some properties of such a ring R. These can be found in [**3**] or else, as indicated, are easy consequences.

(a) For each nonzero prime P of R, there is a "classical" local ring R_P which is the right and left quotient ring of R with respect to the set of regular elements $\mathcal{C}(P) = \{c \in R \mid c+P \text{ is a unit of } R/P\}$. R_P is a prime principal ideal ring. And, as in commutative theory, it is easily checked that R_P is flat as a right or left R-module.

(b) If T is a nonzero ideal of R then T has a unique factorization $T = P_1^{n_1} \cdots P_t^{n_t}$ as a (commutative) product of primes and

$$R/T \cong R/P_1^{n_1} \oplus \cdots \oplus R/P_t^{n_t}.$$

PROOF OF THEOREM. Since every right ideal is a direct summand of an essential right ideal, it will suffice to prove that each essential right ideal A is projective .By (ii), $A \supseteq T$, a nonzero ideal. If T is as described in (b), then $R/A \cong R/B_1 \oplus \cdots \oplus R/B_t$ where each B_i is an essential right ideal of R containing $P_i^{n_i}$. If each B_i were projective, then R/A would have projective dimension 1 and A would be projective. Thus we can assume that $A \supseteq P^n$ for some prime P.

Now consider the ring R_P. Since the elements of $\mathcal{C}(P)$ are units, mod P, they are also units mod P^n. This implies that $R_P/P_P^n \cong R/P^n$,

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and that $R_P/A_P \cong R/A$ as right R/P^n-modules or as right R-modules. Thus there are two exact sequences of right R-modules

$$0 \to A_P \to R_P \to R/A \to 0, \qquad 0 \to A \to R \to R/A \to 0.$$

In the first, A_P is an essential right ideal of the prime principal ideal ring R_P and thus is isomorphic to R_P. Therefore both A_P and R_P are flat right R-modules and so R/A has weak homological dimension 1. Then the second sequence shows that A is flat. But A is finitely generated and R is noetherian and so, by [**2**, p. 122, Exercise 3], A is projective, as required.

References

1. K. Asano, *Zur Arithmetik in Schiefringen*. II, J. Inst. Polytech. Osaka City Univ. Ser. A. Math. **1** (1950), 1–27. MR **12**, 75.

2. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, N. J., 1956. MR **17**, 1040.

3. G. O. Michler, *Asano orders*, Proc. London Math. Soc. **19** (1969), 421–443.

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