A CLASSIFICATION OF IMMERSED KNOTS

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This paper in a sense completes the study begun in [6] and [3]. We shall make extensive use of the notation and results of [6] and shall show that the group of immersed homotopy n-spheres in m-space fits naturally into exact sequences similar to those of [6]. Alternatively, one can look on this paper as giving geometric meaning to the groups $\pi_n G$ and $\pi_n(G, SO_q)$. (See also [4]. For additional information about immersions see §4 of [3].) In §1 we give the necessary definitions; the main results are stated in §2. We would like to thank the referee for some useful observations.

1. **Preliminaries.** Throughout this paper all manifolds will be C^∞, compact, and oriented. All maps are transverse to any boundaries and take boundaries to boundaries. If $f: M^n \to W^m$ is an immersion, we orient the normal bundle of f, ν_f, by the equation

$$\tau_M + \nu_f = f^*\tau_W,$$

where τ_M, τ_W are the tangent bundles of M, W, respectively. The boundary of M, ∂M, is oriented by the equation

$$\xi + \tau_{\partial M} = \tau_M \mid \partial M,$$

where ξ is a line bundle of vectors orthogonal to ∂M with the vectors pointing out from M oriented positively. $-M$ denotes M with the negative orientation. Given a vector bundle η over M we shall always identify M with the zero-section of η; also, we do not distinguish between the normal disk bundle of a submanifold and a tubular neighborhood.

Next, we say that $(f, \mathfrak{F}): M^n \to W^m$ is a framed immersion if $f: M \to W$ is an immersion and $\mathfrak{F}$ is a framing of ν_f, i.e., $\mathfrak{F} = (f_1, \cdots, f_{m-n})$ is an ordered collection of orthogonal vector fields of ν_f (this is compatible with [6] in case f is an imbedding). Now let M^n and W^m be closed manifolds (boundaries are excluded only for simplicity). Two immersions $f, g: M \to W$ are h-cobordant if there is an h-cobordism V^{n+1} with $\partial V = M \cup -M$ and an immersion $H: V \to W \times [0, 1]$ with $H \mid M = f \times 0$ and $H \mid -M = g \times 1$. Two framed immersions $(f, \mathfrak{F})$, $(g, \mathcal{G}): M \to W$ are h-cobordant if there is an h-cobordism V^{n+1} with $\partial V = M \cup -M$ and a framed immersion $(H, \mathcal{H}): V \to W \times [0, 1]$ with $(H, \mathcal{H}) \mid M = (f, \mathfrak{F}) \times 0$ and $(H, \mathcal{H}) \mid -M = (g, \mathcal{G}) \times 1$. We use $[M, f]$, $[M, f, \mathfrak{F}]$ to denote the h-cobordism class of f, $(f, \mathfrak{F})$, respectively—

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the range W will always be clear from the context. Also, let us recall the notion of regular homotopy between immersions (see [5]). Two framed immersions $(f, \mathfrak{F})$, $(g, \mathfrak{G}): M \to W$ will be called regularly homotopic if there is a 1-parameter family of framed immersions $(H_t, \mathfrak{H}_t): M \to W$ with $(H_0, \mathfrak{H}_0) = (f, \mathfrak{F})$ and $(H_1, \mathfrak{H}_1) = (g, \mathfrak{G})$.

As usual, D^n will be the closed unit ball in Euclidean n-space $\boldsymbol{R}^n$ with the natural orientation and $S^{n-1} = \partial D^n$. G_n denotes the H-space of maps of $S^{n-1} \to S^{n-1}$ of degree $+1$ and SO_n will be the subspace of orthogonal maps. The natural inclusion $\boldsymbol{R}^n \subseteq \boldsymbol{R}^{n+1}$ gives rise to inclusions $D^n \subseteq D^{n+1}$, $G_n \subseteq G_{n+1}$, $SO_n \subseteq SO_{n+1}$. Let $G = \lim_{n\to\infty} G_n$ and $SO = \lim_{n\to\infty} SO_n$.

2. **The exact sequences.** For the remainder of this paper we assume that $n \geqq 5$ and $k = m - n \geqq 3$. We define $I^{m,n}$ to be the set of h-cobordism classes of immersed homotopy n-spheres in S^m and $I_f^{m,n}$ to be the set of h-cobordism classes of framed immersed homotopy n-spheres in S^m. There is an obvious operation of connected sum which makes $I^{m,n}$ and $I_f^{m,n}$ into abelian groups (see [2, §1.3 and §1.4]). Using [7] it is easy to see that $[\Sigma^n, f] = 0 \in I^{m,n}$ (or $[\Sigma^n, f, \mathfrak{F}] = 0 \in I_f^{m,n}$) if and only if Σ^n is the boundary of an $(n+1)$-disk U^{n+1} and there is an immersion $H: U^{n+1} \to D^{m+1}$ (or framed immersion $(H, \mathfrak{H}): U^{n+1} \to D^{m+1}$) so that $H|\Sigma = f$ (or $(H, \mathfrak{H})|\Sigma = (f, \mathfrak{F})$).

Define groups P_n as follows:

$$\begin{aligned} P_n &= Z, & n &\equiv 0 \pmod 4) \\ &= Z_2, & n &\equiv 2 \pmod 4) \\ &= 0, & n &\text{ odd.} \end{aligned}$$

In §3 we shall define homomorphisms $\bar{\omega}_i$, $\bar{\phi}_i$, and $\bar{\partial}_i$ making the following sequences exact:

$$(1)_k \quad \cdots \to \pi_n SO_k \xrightarrow{\bar{\omega}_1} I_f^{m,n} \xrightarrow{\bar{\phi}_1} I^{m,n} \xrightarrow{\bar{\partial}_1} \pi_{n-1} SO_k \xrightarrow{\bar{\omega}_1} I_f^{m-1,n-1} \to \cdots$$

$$(2)_k \quad \cdots \to I_f^{m,n} \xrightarrow{\bar{\omega}_2} \pi_n G \xrightarrow{\bar{\phi}_2} P_n \xrightarrow{\bar{\partial}_2} I_f^{m-1,n-1} \xrightarrow{\bar{\omega}_2} \pi_{n-1} G \to \cdots$$

$$(3)_k \quad \cdots \to I^{m,n} \xrightarrow{\bar{\omega}_3} \pi_n(G, SO_k) \xrightarrow{\bar{\phi}_3} P_n \xrightarrow{\bar{\partial}_3} I^{m-1,n-1} \xrightarrow{\bar{\omega}_3} \pi_{n-1}(G, SO_k) \to \cdots.$$

We also get a commutative (up to sign) diagram:

$$(4)_k \quad \begin{tikzcd} & \pi_n SO_k \arrow[dr] \arrow[rr] & & \pi_n G \arrow[rr] \arrow[dr] & & P_n \\ \pi_{n+1}(G, SO_k) \arrow[ur] \arrow[dr] & & I_f^{m,n} \arrow[ur] \arrow[dr] & & \pi_n(G, SO_k) \arrow[ur] \arrow[dr] & \\ & P_{n+1} \arrow[ur] \arrow[rr] & & I^{m,n} \arrow[ur] \arrow[rr] & & \pi_{n-1} SO_k \end{tikzcd}$$

Next, let $Im^{m,n}$ ($Im_f^{m,n}$) be the group of regular homotopy classes of (framed) immersions of S^n in S^m. The group operation is again the connected sum in both cases. Then it follows from [5] that there are natural isomorphisms $Im^{m,n} \approx \pi_n V_{m,n} \approx \pi_n(SO, SO_k)$ and $Im_f^{m,n} \approx \pi_n SO_m \approx \pi_n SO$. Finally, we also consider the groups C_n^k and FC_n^k of isotopy classes of imbedding, respectively, framed imbeddings, of S^n in S^m and the groups $\theta^{m,n}$ and $\theta_f^{m,n}$ of h-cobordism classes of imbedded, respectively, framed imbedded, homotopy n-spheres in S^m. Collecting the results of [3], [6], and this paper we get a great many interrelated exact sequences which we shall not bother to write out here. In addition, there are natural suspension maps of the sequences $(l)_k \xrightarrow{S} (l)_{k+N}$, $l=1, 2, 3$ and $N \geqq 0$, where we take the rear extensions of framings as described in [6, §1.2].

We shall display two interesting diagrams of exact sequences which are derived from standard diagram chasing:

$$(5)_k \qquad \begin{tikzcd}
 & & \pi_{n+1}(G, G_k) \arrow[dl] \arrow[dr] & & \\
\theta^{n+1} \arrow[r] \arrow[d, equal] & C_n^k \arrow[rr] \arrow[d] & & \theta^{m,n} \arrow[r] \arrow[d] & \theta^n \arrow[d, equal] \\
I^{n+1} \arrow[r] & Im^{m,n} \arrow[rr] \arrow[dr] & & I^{m,n} \arrow[r] \arrow[dl] & I^n \\
 & & \pi_n(G, G_k) & &
\end{tikzcd}$$

$$(6)_k \qquad \begin{tikzcd}
 & I^{m,n} \arrow[r] & I^{m+1,n} \arrow[dr] & \\
\pi_n S^k = \pi_n(SO_{k+1}, SO_k) \arrow[ur] \arrow[dr] & & & \pi_{n-1}(SO_{k+1}, SO_k) = \pi_{n-1} S^k, \\
 & Im^{m,n} \arrow[r] \arrow[uu] & Im^{m+1,n} \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

where $\theta^n = \theta^{m+N,n}$ and $I^n = I^{m+N,n}$ for $N > n$. Also, observe that $I_f^{m,n}$ is independent of m.

In conclusion, we point out that this paper could have been extended to the case of "relative" immersed knots à la [1].

3. **The maps** $\bar{\omega}_i$, $\bar{\Phi}_i$, $\bar{\partial}_i$. In this section we shall define the maps $\bar{\omega}_i$, $\bar{\Phi}_i$, and $\bar{\partial}_i$. Let ω_i, ϕ_i, and ∂_i be defined as in [6] and let $j_1 : \theta^{m,n} \to I^{m,n}$, $j_2 : \theta_f^{m,n} \to I_f^{m,n}$ be the obvious maps which assign to each h-cobordism class of imbeddings, respectively, framed imbeddings, its corresponding h-cobordism class as an immersion, respectively, framed immer-

sion. (It should be clear that $\theta^{m,n}$ and $\theta_f^{m,n}$ can be so interpreted. See also [1].)

Define

$$\bar{\omega}_1 = j_2\omega_1, \qquad \bar{\phi}_2 = \phi_2,$$
$$\bar{\phi}_3 = \text{composition of } \pi_n(G, SO_k) \to \pi_n(G, SO) \xrightarrow{\phi_3} P_n,$$
$$\bar{\partial}_2 = j_2\partial_2, \qquad \bar{\partial}_3 = j_1\partial_3.$$

Let $[\Sigma^n, f, \mathfrak{F}]\in I_f^{m,n}$ and $[N^n, g]\in I^{m,n}$. Define

$$\bar{\phi}_1([\Sigma^n, f, \mathfrak{F}]) = [\Sigma^n, f] \in I^{m,n},$$
$$\bar{\partial}_1([N^n, g]) = \nu_g \in \pi_{n-1}SO_k,$$

where we identify the normal bundle ν_g with the element in $\pi_{n-1}SO_k$ that it determines by the classification theorem of bundles over spheres.

Finally we come to the maps $\bar{\omega}_2$ and $\bar{\omega}_3$ which, together with the exactness of $(2)_k$ at $\pi_n G$ and $(3)_k$ at $\pi_n(G, SO_k)$, are really the heart of this paper.

LEMMA 3.1. *$\theta_f^{N,n} \overset{j_2}{\approx} I_f^{N,n}$, for $N\geqq 2n+1$.*

PROOF. Now all homotopy n-spheres Σ^n imbed in S^N and all immersions $f:\Sigma\to S^N$ are regularly homotopic by [5]. A regular homotopy also carries along the framing. Therefore j_2 is onto. That j_2 is one-to-one follows from the fact that we may approximate regular homotopies by imbeddings using Whitney's theorem.

Thus we can define

$$\bar{\omega}_2 = \text{composition of } I_f^{m,n} \xrightarrow{S} I_f^{N,n} \xrightarrow{j_2^{-1}} \theta_f^{N,n} \xrightarrow{\omega_2} \pi_n G,$$

where $N\geqq 2n+1$.

Now let $[\Sigma^n, f]\in I^{m,n}$. It is well known (see [7]) that Σ^n is obtained by glueing two n-disks D_0 and D_1 together via a diffeomorphism of their boundaries. We may then take framings $\mathfrak{F}_i$ of $\nu_f|D_i$, $i=0, 1$, and let $\alpha_i: D_i\times D^k\to\nu_f|D_i$ be imbeddings satisfying $\alpha_i(x, 0)=x$, $x\in D_i$ and $d\alpha_i(\mathcal{E}_i)=\mathfrak{F}_i$, where $\mathcal{E}_i$ is the pull-back by the projection $D_i\times D^k\to D^k$ of a positive frame at $0\in D^k$. We can arrange it so that $\alpha_1(x, y)=\alpha_0(x, \mu(x)y)$, $x\in\partial D_0$, $y\in D^k$, for some $\mu:\partial D_0\to SO_k$. (Compare [6, §3.3].) We shall use the notation $\mathfrak{F}_1|\partial D_0=\mu\mathfrak{F}_0|\partial D_0$ to describe this situation. Now assume that $f|D_0$ is an imbedding and that $f(\Sigma-D_0)$ does not meet $f(D_0)$. Let f' be the composition of $\Sigma\xrightarrow{f} S^m\to S^{m+N}$, for some $N>n$, and let $\mathfrak{F}_i'$ be the framing of $\nu_{f'}|D_i$ which is the rear extension of $\mathfrak{F}_i$. Next, move f' into an imbedding $g:\Sigma\to S^{m+N}$ via a regular homotopy h_t satisfying $h_t|D_0=f'$ and $h_t(\Sigma-D_0)\cap h_t(D_0)=\emptyset$. h_t carries along the framing $\mathfrak{F}_i'$, so that we get a framing $\mathcal{G}_i$ of $\nu_g|D_i$ with $\mathcal{G}_1|\partial D_0=\bar{\mu}\mathcal{G}_0|\partial D_0$, where $\bar{\mu}$ is the composition of $\partial D_0\xrightarrow{\mu}SO_k\to SO_{k+N}$. If we apply the construction of [6, §3.3], to $(\Sigma, g, \mathcal{G}_i)$, we get an element $[\lambda]\in\pi_n(G, SO_k)$, i.e., if we let $u:S^{m+N}-g(\Sigma)\to S^{k+N-1}$ be a homotopy inverse of $y\to\alpha_0(x_0, y)$, $x_0\in D_0$ with $u\alpha_0(x, y)=y$ for all $x\in D_0$, then $\lambda:D_1\to G_{k+N}$ is given by $\lambda(x)(y)=u\alpha_1(x, y)$, $x\in D_1$, $y\in S^{k+N-1}$. Define

$$\bar{\omega}_3([\Sigma, f]) = [\lambda].$$

This finishes the definitions of all the maps and it is easy to see, using [6], that they are well defined homomorphisms.

4. **Exactness.** The proof of exactness of $(1)_k$, $(2)_k$, and $(3)_k$ is very similar to the corresponding proofs given in [6]. In general, the only difference is that here we have immersions instead of imbeddings. Anyone who understands [6] can easily make the appropriate translations. We shall, however, outline a proof of exactness in those places that differ from the corresponding ones in [6]. One essential difference is the fact that any abstract framed surgery can be realized ambiently. Another is that framed immersions of n-spheres in S^m are regularly homotopic to immersions in S^{n+1}.

We first prove exactness at π_nG in $(2)_k$. That $\bar{\Phi}_2\bar{\omega}_2=0$ follows from [6, §5.5], and the definition of $\bar{\omega}_2$ and $\bar{\Phi}_2$. Let $[g]\in\pi_nG$, $g:S^n\to G_N$ and suppose $\bar{\Phi}_2([g])=0$. Define $\bar{g}:S^n\times S^{N-1}\to S^{N-1}$ by $\bar{g}(x, y)=g(x)(y)$, $x\in S^n$, $y\in S^{N-1}$, and let $\Sigma=\bar{g}^{-1}(e)$, $e\in S^{N-1}$. We may assume that Σ is a framed n-submanifold of $S^n\times S^{N-1}\subseteq S^{n+N}$. In fact, since $\bar{\Phi}_2([g])=0$, we may further assume that Σ is a homotopy sphere (see [6, §4.7]). By Theorem 6.4 of [5], Σ is regularly homotopic to a framed immersion (Σ, f) in S^m. Then $\bar{\omega}_2([\Sigma, f])=[g]$.

Next, let us consider exactness at $I_f^{m,n}$. There is no problem in showing that $\bar{\omega}_2\bar{\partial}_2=0$. Suppose $[\Sigma, f, \mathfrak{F}]\in I_f^{m,n}$ and $\bar{\omega}_2([\Sigma, f, \mathfrak{F}])=0$. It follows from the definition of $\bar{\omega}_2$ and the exactness of the Kervaire-Milnor sequence that there is a π-manifold W and a framing of its stable normal bundle so that $\partial W=\Sigma$ and the framing restricted to Σ is essentially a suspension of $\mathfrak{F}$. But then we can use [5] to obtain a framed immersion $(g, \mathcal{G}):W\to S^m$ so that $g|\Sigma=f$ and $\mathcal{G}|\Sigma=\mathfrak{F}$. Define $\gamma=\gamma(W, \mathcal{G})\in P_{n+1}$ as in §4.5 of [6]. Then $\bar{\partial}_2(\gamma)=[\Sigma, f, \mathfrak{F}]$, because using [5] we can allow in the definition of $\bar{\partial}_2$ not only framed imbeddings of W but also framed immersions.

This finishes our discussion of the exactness of $(2)_k$. Alternatively, one could observe first that $I_f^{m,n} \approx I_f^n \approx \theta_f^n$ using [5], so that exactness follows from the exactness of the Kervaire-Milnor sequence.

The commutativity (up to sign) of $(4)_k$ is proved as in [6] and so by [6, §5.3], the exactness of $(3)_k$ will be established once we show that $\bar{\phi}_3\bar{\omega}_3 = 0$. But consider

$$\begin{tikzcd}
I^{m,n} \arrow[r, "\bar{\omega}_3"] \arrow[d, "S"] & \pi_n(G, SO_k) \arrow[dd, "S"] \arrow[ddr, "\bar{\phi}_3"] & \\
I^{m+N,n} & & \\
\theta^{m+N,n} \arrow[u, "j_1"] \arrow[r, "\omega_3"] & \pi_n(G, SO) \arrow[r, "\phi_3"] & P_n.
\end{tikzcd}$$

If $N > n$, then j_1 is an isomorphism (proved similarly to Lemma 3.1), and so $\phi_3\omega_3 = \phi_3\omega_3 j_1^{-1} S = 0$.

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