POSITIVE LENGTH BUT ZERO ANALYTIC CAPACITY

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Let E be a subset of the complex plane. For m positive define

$$A(E, m) = \{f: f \text{ is analytic off a compact subset of } E, f(\infty) = 0, \|f\| \leqq m\}.$$

For $f \in A(E, m)$, the derivative at ∞ of f is its first Laurent coefficient $f'(\infty) = \lim_{z\to\infty} zf(z)$. The analytic capacity of E is defined as

$$\gamma(E) = \sup\{\,|f'(\infty)| : f \in A(E, 1)\}.$$

Of the several uses of analytic capacity we only mention one: that $\gamma(E) = 0$ if and only if every bounded analytic function on the complement of E is constant.

The length, or one-dimensional Hausdorff measure, of a set E is $l(E) = \lim_{\rho\to 0} \Lambda_\rho(E)$, where

$$\Lambda_\rho(E) = \inf\left\{\sum \delta_j : E \subset \bigcup \Delta(a_j, \delta_j),\ \delta_j \leqq \rho\right\}$$

and $\Delta(a_j, \delta_j)$ is the disc $\{|z - a_j| < \delta_j\}$. When E lies on a rectifiable curve, this notion is equivalent to that of the (outer) arc length of E.

A classical theorem of Painlevé is that $\gamma(E) = 0$ whenever $l(E) = 0$. When E lies on a sufficiently smooth curve, $\gamma(E)$ and $l(E)$ can only vanish simultaneously [2]. On the other hand, A. G. Vituškin [3] has given an example of a set E with $l(E) > 0$ but $\gamma(E) = 0$. However Vituškin's proof is quite complicated and contains many typographical errors. We give a simpler counterexample, and compare ours to Vituškin's.

1. **The example.** The example is the planar Cantor set obtained by taking the "corner quarters." Let $K = \bigcap_{n=0}^\infty E_n$ where E_0 is the unit square, E_n consists of 4^n squares of side 4^{-n}, and each component of E_n contains four components of E_{n+1}, these being the four corner squares of side 4^{-n-1}. The components of E_n will be indexed as $E_{n,j}$, $1 \leqq j \leqq 4^n$. Set $K_{n,j} = K \cap E_{n,j}$.

It is obvious that $\Lambda(K) = 2^{-1/2}$.

Our proof that $\gamma(K) = 0$ resembles Vitushkin's argument but is simpler because it takes advantage of the homogeneity of the set K. That is, $K_{n,j}$ is geometrically similar to K, so that one can make linear changes of variable, and so that $\gamma(K_{n,j}) = 4^{-n}\gamma(K)$.

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Let $f \in A(K, 1)$, and assuming $\gamma(K)>0$, suppose $a=f'(\infty)$ is real and positive. For $z \notin K$, let $\Gamma_{n,j}$ be a cycle with winding number one about $K_{n,j}$ but zero about $K-K_{n,j}$ and about z, and set

$$f_{n,j}(z) = \frac{-1}{2\pi i}\int_{\Gamma_{n,j}} \frac{f(w)dw}{w-z}.$$

Lemma 1. (a) $\sum_{j=1}^{4^n} f_{n,j} = f$.

(b) *There is a constant* M *such that* $f_{n,j} \in A(K_{n,j}, M)$.

(c) $|f'_{n,j}(\infty)| \leqq M4^{-n}\gamma(K)$.

Proof. The Cauchy integral theorem yields (a) as well as the fact that $f_{n,j}$ is analytic off $K_{n,j}$. If V is a square concentric with $E_{n,j}$ but three times as large, then for z near $K_{n,j}$ we have

$$f(z) - f_{n,j}(z) = \frac{1}{2\pi i}\int_{\partial V} \frac{f(w)dw}{w-z},$$

so that $|f_{n,j}(x)| \leqq 1+6/\pi$. (c) follows directly from (b).

Set $a_{n,j}=f'_{n,j}(\infty)$.

Lemma 2. *Let*

$$h_{n,j}(z) = a_{n,j}4^{2n}\iint_{E_{n,j}} \frac{dxdy}{x+iy-z}$$

for $z \notin E_{n,j}$, *and set* $h_n = \sum_{j=1}^{4^n} h_{n,j}$. *Then the* h_n *are uniformly bounded.*

Proof. A well-known estimate using polar coordinates and part (c) of Lemma 1 yield the estimate $|h_{n,j}(z)| \leqq M\gamma(K)\cdot 2^{1/2}\pi = M_1$. Also $h'_{n,j}(\infty) = -a_{n,j}$. If $g_{n,j}=f_{n,j}+h_{n,j}$; then $|g_{n,j}| \leqq M+M_1=M_2$ and $g_{n,j}$ vanishes twice at ∞. Two applications of Schwarz's lemma then give

$$|g_{n,j}(z)| \leqq \frac{M_2 4^{-2n}}{(\operatorname{dist}(z, E_{n,j}))^2}.$$

As $f=\sum f_{n,j}$, we must show $\sum_{j=1}^{4^n} g_{n,j}$ is uniformly bounded. Now for z close to E_{n,j_0} we have

$$\sup_{z\notin E_n} \sum_j |g_{n,j}(z)| \leqq \sup_{z\notin E_n}\left(1 + \sum_{j\neq j_0}\frac{M_2 4^{-2n}}{(\operatorname{dist}(z, E_{n,j}))^2}\right) = B_n.$$

To estimate B_n, compare the 4^{n-1} terms corresponding to the 4^{n-1} squares in the same component of E_1 as z with the supremum B_{n-1}, and estimate the remaining terms by $4M_2 4^{-2n-1}$. This gives $B_n \leqq B_{n-1} + 3\cdot 4^{n-1}\cdot 4M_2 4^{-2(n-1)}$, so that $\lim B_n < \infty$, and the h_n are uniformly bounded.

Actually one can show that $f=-\lim_n h_n$, but we will not need this. We finish the proof by showing that the case $a_{n,j}=a4^{-n}$ for all j is impossible, but that otherwise Lemma 1(c) is contradicted.

LEMMA 3. *For some n and j, $a_{n,j}\neq a4^{-n}$.*

PROOF. Assume $a_{n,j}=a4^{-n}$ for all n and j. Then $A_n=a^{-1}\,\mathrm{Real}(h_n(0))$ is bounded. Now

$$A_n = 4^n \sum_{j=1}^{4^n} \iint_{E_{n,j}} \frac{x\,dxdy}{x^2+y^2}.$$

But

$$4^n \sum_{E_{n,j}\subset\{x>\frac{1}{2}\}} \iint_{E_{n,j}} \frac{x\,dxdy}{x^2+y^2} \geqq 4^{-2}$$

while the sum of the 4^{n-1} lower left integrals is A_{n-1}. Thus $A_n\geqq A_{n-1}+4^{-2}$, a contradiction.

LEMMA 4. *For any $\epsilon>0$ and any $M>0$, there exists $\delta>0$ such that for any $f\in A(K, M)$ with $|f'(\infty)|\geqq\epsilon$, we have*

$$\sup_{n,j} 4^n\,|a_{n,j}| \geqq (1+\delta)\,|f'(\infty)|.$$

PROOF. Suppose not. Then for $\delta_k\searrow 0$ there exists $f_k\in A(K, M)$ with $|f_k'(\infty)|\geqq\epsilon$ such that (giving $a_{n,j}^{(k)}$ its obvious meaning)

$$|a_{n,j}^{(k)}| \leqq 4^{-n}(1+\delta_k)\,|f_k'(\infty)|.$$

A subsequence of $\{f_k\}$ then converges to $f\in A(K, M)$ with $|f'(\infty)|\geqq\epsilon$ and with $|a_{n,j}|\leqq 4^{-n}|f'(\infty)|$. This means $a_{n,j}=4^{-n}f'(\infty)$, which is impossible by Lemma 3.

Finally, to show $\gamma(K)=0$, let $f\in A(K, 1)$ with $a=f'(\infty)>0$. Choose n_1 and j_1 such that by Lemma 4 (with $\epsilon=a$ and M as in Lemma 1), $|a_{n_1,j_1}|\geqq a(1+\delta)4^{-n_1}$. Since K_{n_1,j_1} is geometrically similar to K we can apply Lemma 4 to f_{n_1,j_1}. Continuing, we obtain a sequence (n_k, j_k) with $|a_{n_k,j_k}|\geqq a(1+\delta)^k 4^{-n_k}$. This contradicts Lemma 1(c).

2. **Comparison with Vitushkin's example.** Take a nondecreasing sequence of positive integers n_j with $n_1\geqq 2$. Set $E_0=[0, 1]$, $E_1=\bigcup_{k=1}^{n_1}(\{1/k\}\times[0, 1/n_1])$. Obtain E_{k+1} from E_k by repeating this process with each interval in E_k but using n_{k+1}. Set $E=\lim_k E_k$. For any choice of n_j, $l(E)$ is positive. Vitushkin [3] shows that $\gamma(E)=0$

if $\lim_{j\to\infty} n_j = \infty$. Setting $r_k = \prod_{j=1}^{k} n_j^{-1}$, we see that E_k consists of pairwise disjoint intervals of length r_k. Replacing each interval by a rectangle of sides r_k and r_{k+1}, we have $E = \bigcap E_k$. Then the reasoning of §1 above shows that $\gamma(E) = 0$ if $n_j = m$ for all sufficiently large indices j.

3. **Remarks.** It is interesting to compare our example with the known results for similar Cantor sets. Let C_r $(0 < r < 1)$ be the Cantor set on $[0, 1]$ obtained by removing $r$ths, and let $K_r = C_r \times C_r$. Thus our set is $K_{1/2}$. For $r > 1/2$, $l(K_r) = 0$, and thus $\gamma(K_r) = 0$. For $r < 1/2$, Denjoy [1] proved that $\gamma(K_r) > 0$. Indeed, Denjoy constructed a function in $A(K_r, 1)$ which extended continuously to the entire plane. In other words, if $r < 1/2$, K_r has positive continuous analytic capacity: $\alpha(K_r) > 0$. It was also known [4], that $\alpha(K_{1/2}) = 0$, because $l(K_{1/2}) < \infty$. Thus our example completes the study of analytic capacity for such Cantor sets.

References

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