ON THE INVERSE OF AN INTEGRAL OPERATOR¹

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We wish to consider the integral equation

$$(1) \quad f(x) = \frac{i}{2} \int_{-1}^1 H_0^{(1)}(k|x-t|) \phi(t) dt.$$

Here $H_0^{(1)}$ denotes the zero order Hankel function of the first kind. k is a nonzero constant with $\operatorname{Re} k \geq 0$, $\operatorname{Im} k \geq 0$. Recall that for small r we have

$$(2) \quad \frac{i}{2} H_0^{(1)}(kr) = \frac{1}{\pi} \log \frac{1}{r} + h(r)$$

where $h(r)$ and $h'(r)$ are finite at $r=0$. The equation (1) arises in connection with the solution of the reduced wave equation in the plane slit along the x -axis from -1 to $+1$ [1].

In [1] the following result was proven: Let h denote the class of complex functions ϕ which are Hölder continuous in a neighborhood of each point of $(-1, 1)$ and further satisfy the condition that near $x=1$, $|\phi(x)| \leq \kappa/(1-x)^\alpha$, $0 \leq \alpha < 1$ and near $x=-1$, $|\phi(x)| \leq \kappa/(1+x)^\alpha$. Then given $f(x)$ such that f' is Hölder continuous, equation (1) has a unique solution, $\phi \in h$. In this paper we will consider equation (1) as a mapping from one Hilbert space into another. We will show that if the domain and range spaces are defined appropriately the integral operator in (1) becomes a one-to-one continuous mapping of one Hilbert space onto another and hence by Banach's open mapping theorem has a continuous inverse. It will be shown that if f is sufficiently smooth, the solutions found here coincide with those found in [1].

Let $p(t) = (1-t^2)^{-1/2}$, $-1 < t < 1$ and $q(t) = (1-t^2)^{1/2} = 1/p(t)$, $-1 < t < 1$. We define three spaces:

$$L_2(p) = \left\{ f \left| \int_{-1}^1 |f|^2 (1-t^2)^{-1/2} dt < \infty \right. \right\};$$

$$L_2(q) = \left\{ f \left| \int_{-1}^1 |f|^2 (1-t^2)^{1/2} dt < \infty \right. \right\};$$

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$W_2^1(q) = \{f \mid f \text{ is absolutely continuous on } [-1, 1] \text{ and } f' \text{ (which exists a.e. with respect to Lebesgue measure)} \in L_2(q)\}.$

If in $L_2(p)$ we define $\|f\|_{L_2(p)}^2 = \int_{-1}^1 |f|^2 (1-t^2)^{-1/2} dt$ and in $L_2(q)$ we define $\|f\|_{L_2(q)}^2 = \int_{-1}^1 |f|^2 (1-t^2)^{1/2} dt$ then these spaces are Hilbert spaces. In $W_2^1(q)$ we define

$$\|f\|_{W_2^1(q)}^2 = \|f\|_{L_2(q)}^2 + \|f'\|_{L_2(q)}^2.$$

We then have

THEOREM 1. *Under the above norm $W_2^1(q)$ is a Hilbert space.*

PROOF. We first note that $L_2(q) \subset L_1(-1, 1)$ (the usual class of functions integrable over $(-1, 1)$ with respect to Lebesgue measure) and the injection is continuous. This can be seen by using the Schwarz inequality in $L_2(q)$.

Now suppose $\{f_n\}$ is a Cauchy sequence in $W_2^1(q)$. In particular $\{f'_n\}$ is Cauchy in $L_2(q)$. Thus there exists $g \in L_2(q)$ such that $\|f'_n - g\|_{L_2(q)} \rightarrow 0$. By the above remark $f'_n, g \in L_1(-1, 1)$. Thus

$$f_n(x) = f_n(-1) + \int_{-1}^x f'_n(t) dt.$$

Hence

$$f_n(-1) - f_m(-1) = f_n(x) - f_m(x) - \int_{-1}^x (f'_n(t) - f'_m(t)) dt.$$

Thus

$$|f_n(-1) - f_m(-1)|^2 \leq 2 |f_n(x) - f_m(x)|^2 + 2 \|f'_n - f'_m\|_1^2.$$

If we multiply by $q(t)$ and integrate from -1 to 1 we find

$$\frac{\pi}{2} |f_n(-1) - f_m(-1)|^2 \leq 2 \|f_n - f_m\|_{L_2(q)}^2 + \pi \|f'_n - f'_m\|_1^2.$$

Thus

$$|f_n(-1) - f_m(-1)|^2 \leq \frac{4}{\pi} \|f_n - f_m\|_{L_2(q)}^2 + 2\pi \|f'_n - f'_m\|_{L_2(q)}^2 \rightarrow 0$$

as $m, n \rightarrow \infty$. Thus $f_n(-1) \rightarrow C$ as $n \rightarrow \infty$. Let $f(x) = C + \int_{-1}^x g(t) dt$. Then f is absolutely continuous and

$$f(x) - f_n(x) = C - f_n(-1) + \int_{-1}^x (g(t) - f'_n(t)) dt.$$

Thus

$$|f(x) - f_n(x)|^2 \leq 2 |C - f_n(-1)|^2 + 2 \|g - f'_n\|_1^2.$$

As above it then follows that

$$\|f - f_n\|_{L_2(q)}^2 \leq \pi |C - f_n(-1)|^2 + 2\pi \|g - f'_n\|_{L_2(q)}^2 \rightarrow 0$$

as $n \rightarrow \infty$. Thus $\|f_n - f\|_{W_2^1(q)} \rightarrow 0$ as $n \rightarrow \infty$. ■

We now consider the operator defined by (1). Let

$$(3) \quad \psi(x) = \frac{i}{2} \int_{-1}^1 H_0^{(1)}(k|x-t|) \phi(t) dt = (L\phi)(x).$$

As is pointed out in [1] if ϕ is Hölder continuous we may differentiate under the integral sign and obtain (in view of (2)):

$$(4) \quad \psi'(x) = \frac{1}{\pi} \int_{-1}^1 \frac{\phi(t) dt}{x-t} + \int_{-1}^1 k(t, x) \phi(t) dt$$

where the first term must be taken as a Cauchy Principal Value and in the second term $k(t, x)$ is a continuous kernel.

We now consider (4) as an equation in $L_2(q)$. Let $F: L_2(q) \rightarrow L_2(p)$ be defined by $(Ff)(t) = (1-t^2)^{1/2} f(t)$. Then F is an isometry of $L_2(q)$ onto $L_2(p)$. Define an operator T by

$$(5) \quad Tg = \frac{1}{\pi} \int_{-1}^1 \frac{g(t)}{x-t} \cdot \frac{1}{(1-t^2)^{1/2}} dt.$$

Then we have the following theorem [2].

THEOREM 2. *The operator defined by (5) is a continuous mapping from $L_2(p)$ onto $L_2(q)$. Its null space is one dimensional and is spanned by the function $g(x) \equiv 1$. Further the restriction, T_0 , of T to the orthogonal complement $H(p)$ of this null space is an isometry of $H(p)$ onto $L_2(q)$ with inverse mapping*

$$T_0^{-1}h = \frac{1}{\pi} \int_{-1}^1 \frac{h(t)}{t-x} (1-t^2)^{1/2} dt.$$

Thus the mapping $\pi^{-1} \int_{-1}^1 \phi(t)/(x-t) dt$ can be written as $TF\phi$. We see that it maps $L_2(q)$ continuously onto $L_2(q)$ with a one dimensional null space spanned by $p(t) = (1-t^2)^{-1/2}$. We recall the definition of

the index of an operator S from one linear space X to another linear space Y . Suppose S has a finite dimensional null space $N(S)$, $\dim N(S) = \alpha(S)$, and that the range of S , $R(S)$, has finite codimension. Then $\text{codim } R(S) = \dim Y/R(S) = \beta(S)$. (In which case S is said to be a Fredholm operator.) The integer $i(S) = \alpha(S) - \beta(S)$ is called the *index* of the operator S . Thus we have that TF is a Fredholm operator with $\alpha(TF) = 1$, $\beta(TF) = 0$. Thus $i(TF) = 1$. Since $k(t, x)$ is continuous so that

$$\int_{-1}^1 \int_{-1}^1 |k(t, x)|^2 (1 - t^2)^{-1/2} (1 - x^2)^{1/2} dx dt < \infty,$$

$\int_{-1}^1 k(t, x) \phi(t) dt$ represents a compact operator, K_0 , from $L_2(q)$ into $L_2(q)$. Now the operator TF admits a *left regularization* [3], i.e. there exists a linear bounded operator Q mapping $L_2(q)$ into $L_2(q)$ such that $Q(TF) = I + K$ where I is the identity in $L_2(q)$ and K is a compact operator (we take $Q = F^{-1}T_0^{-1}$. Then $K = -P_0$ where P_0 is the projection onto the space spanned by $p(t) = 1/(1 - t^2)^{1/2}$. We then note:

THEOREM 3 [3]. *If a bounded operator A admits a left regularization and has finite index and K is any compact operator we have $i(A + K) = i(A)$.*

Hence we conclude that the mapping defined by the right-hand side of (4) is a continuous mapping of $L_2(q)$ into $L_2(q)$ with index equal to 1.

We return now to the operator L defined by (3). We have $\int_{-1}^1 \int_{-1}^1 |H_0^{(1)}(k|x-t|)|^2 (1 - t^2)^{-1/2} (1 - x^2)^{1/2} dt dx < \infty$. Thus L is a continuous (compact) operator from $L_2(q)$ into $L_2(q)$.

THEOREM 4. *The operator L maps $L_2(q)$ into $W_2^1(q)$.*

PROOF. Given $\phi \in L_2(q)$. Let

$$\psi = L\phi, \quad \chi = TF\phi + K_0\phi.$$

Let $\{\phi_n\}$ be a sequence of Hölder continuous functions $\ni \|\phi_n - \phi\|_{L_2(q)} \rightarrow 0$. Let $\psi_n = L\phi_n$.

Then we know that ψ_n is differentiable on $(-1, 1)$ and $\psi_n' = TF\phi_n + K_0\phi_n$. By continuity of the mappings L and $TF + K_0$ we see that $\{\psi_n\}$ and $\{\psi_n'\}$ are Cauchy sequences in $L_2(q)$ i.e. $\{\psi_n\}$ is a Cauchy sequence in $W_2^1(q)$. By Theorem 1 $\exists \psi_0 \in W_2^1(q) \ni \|\psi_n - \psi_0\|_{W_2^1(q)} \rightarrow 0$. Hence $\|\psi_n - \psi_0\|_{L_2(q)} \rightarrow 0$ but $\psi_n \rightarrow \psi$ in $L_2(q)$. Thus $\psi = \psi_0$ a.e. In fact $\psi \equiv \psi_0$ since ψ can easily be shown to be continuous and ψ_0 is absolutely continuous. Also $\chi = \psi_0'$ a.e. Hence the theorem is proven.

THEOREM 5. *The operator L is a one-to-one map of $L_2(q)$ onto $W_2^1(q)$.*

PROOF. Let $f \in W_2^1(q)$ and consider the equation in $L_2(q)$

$$(6) \quad f' = (TF + K_0)\phi.$$

We know that the index of $(TF + K_0)$ is 1. Thus $\alpha(TF + K_0) \geq 1$. Let $\phi_0 \in L_2(q)$ satisfy the equation

$$(7) \quad TF\phi_0 + K_0\phi_0 = 0.$$

Recall that

$$K_0\phi_0 = \int_{-1}^1 k(t, x)\phi(t)dt, \quad k(t, x) = h'(|t - x|) \sim (t - x) \log |t - x|.$$

Now $k(t, x)$ is Hölder continuous in x uniformly in t (see [4, p. 17]). Thus an easy argument shows that if $\phi_0 \in L_2(q)$, $K_0\phi_0$ is Hölder continuous. Thus applying the operator $F^{-1}T_0^{-1}$ we see that

$$\phi_0(x) = \frac{1}{\pi} \frac{1}{(1 - x^2)^{1/2}} \int_{-1}^1 \frac{(K_0\phi_0)(t)}{t - x} (1 - t^2)^{1/2} dt + \frac{C}{(1 - t^2)^{1/2}}$$

but from this we see that $\phi_0 \in h$. Hence all solutions of (7) in $L_2(q)$ are at the same time in h . Hence applying arguments as in [1] we see that there exists exactly 1 linearly independent solution of (6) in $L_2(q)$, say ϕ_0 . Further $L\phi_0 \equiv C_0$ where C_0 is a nonzero constant. Thus $\alpha(TF + K_0) = 1$, $\beta(TF + K_0) = 0$, i.e. $TF + K_0$ is onto. Let ϕ_f be a solution of (6). Then we consider the function $f - L\phi_f$. This is a function in $W_2^1(q)$ with derivative $f' - (TF + K_0)\phi_f = 0$ a.e. Thus $f - L\phi_f = C_f$ where C_f is a definite constant. Thus $\phi^* = \phi_f + (C_f/C_0)\phi_0$ satisfies $L\phi^* = f$. The above argument shows that this solution is unique. ■

THEOREM 6. *L^{-1} is a continuous mapping from $W_2^1(q)$ onto $L_2(q)$.*

PROOF. Apply Banach's open mapping theorem.

Finally we note that if f' is Hölder continuous and ϕ is the solution of $L\phi = f$ we have $(TF + K)\phi = f'$ and applying the operator $F^{-1}T_0^{-1}$ as is the proof of Theorem 5 we again see that $\phi \in h$. Hence the solutions found here coincide with those found in [1].

ADDENDUM. It has recently come to my attention that similar results can be obtained if we consider the integral operator as mapping L_p onto W_p^1 with $1 < p < 2$. Here W_p^1 is the usual Sobolev space. The key step (the analogue of Theorem 2) is supplied by Theorem 2 of [5].

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