ANOTHER EXOTIC ACTION ON E^n

GLEN E. BREDON

In [1, p. 353] Mostert asks whether, for a torus T acting on E^n , the stationary set $F(G, E^n)$ is connected for each subgroup G of T . This is asked in connection with a conjecture of the Hsiangs, but it would be of considerable interest in itself if it were true. However, it is not too difficult to construct counterexamples. One can, for example, find an action of the circle group on D^6 (or E^6) such that the subgroup Z_6 has fixed point set $T^2 + D^2$ (resp. $T^2 + E^2$). It is slightly more difficult to construct such examples in such a way that the action extends to that of some semisimple group.

In this note we shall show how to construct a smooth action of $SO(3)$ on a disk D^n , with linear action on the boundary, such that a subgroup Z_6 has a disconnected fixed point set. The method applies to other situations as well.

We start with a known smooth action of Z_6 on S^5 with fixed set the disjoint union of two circles; see [1, p. 60]. We denote this action by θ . Let x be a fixed point of θ and let θ_x be the tangential representation of Z_6 at x . (In the example cited, this is the representation

$$\theta_x(A) = \begin{bmatrix} A^2 & & \\ & A^3 & \\ & & 1 \end{bmatrix} \in SO(5)$$

where A is a generator of $Z_6 \subset SO(2)$.)

Now let α be a representation of $SO(3)$ such that there is a point z whose isotropy group is Z_6 and such that the slice representation at z has the form $\theta_x \oplus \beta$ for some representation β of Z_6 (the slice representation is the induced representation of the isotropy group on the normal plane to the orbit). This can always be done and, in fact, the 16-dimensional representation $\alpha = \rho_{13} \oplus \rho_3$ of $SO(3)$ has this property.

Let the dimension of β be r (if $\alpha = \rho_{13} \oplus \rho_3$ then $r = 8$) and let S^r be the unit sphere in the representation $\beta \oplus 1$. Let Z_6 act on $S^5 \times S^r$ by θ on the first factor and $\beta \oplus 1$ on the second, and let $y \in S^r$ be a fixed point so that (x, y) is a fixed point of this action. Let $x' \neq x$ also be fixed for θ and let D^5 be a disk about x' in S^5 on which θ is linear. Clearly we may perform the equivariant surgery of removing $D^5 \times S^r$ and pasting in $S^4 \times D^{r+1}$. This gives an action of Z_6 on S^{r+5} . If

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$p \in S^{r+5}$ corresponds to (x, y) then the representation of Z_6 at p is just $\theta_x \oplus \beta$. If S^k is the fixed set of $\beta \oplus 1$ on S^r ($k=2$ in the specific case mentioned) then $F(Z_6, S^{r+5}) \approx (S^1 \times S^k) + S^{k+1}$ and p can be taken to be in either component. We now remove a disk from S^{r+5} about p . This gives an action of Z_6 on D^{r+5} whose fixed point set is $(S^1 \times S^k) + D^{k+1}$ (or $(S^1 \times S^k - E^{k+1}) + S^{k+1}$ as one wishes) and, moreover, such that the action on the boundary S^{r+4} is just the linear action $\theta_x \oplus \beta$.

Now if D_1^{r+5} is a normal disk to the orbit at z of the representation α (so that the isotropy group Z_6 acts on D_1^{r+5} via $\theta_x \oplus \beta$) then a tubular neighborhood of the orbit $SO(3)(x)$ has the form

$$SO(3) \times_{Z_6} D_1^{r+5}$$

where Z_6 acts on $SO(3)$ by right translation and on D_1^{r+5} by $\theta_x \oplus \beta$ and $SO(3)$ acts by left translation of the left hand factor. Since Z_6 acts on the boundary of D^{r+5} in the same way as it acts on that of D_1^{r+1} , we can equivariantly replace this tubular neighborhood by

$$SO(3) \times_{Z_6} D^{r+5}$$

thereby obtaining an action λ of $SO(3)$ on E^{r+8} .

Now the fixed point set of Z_6 on $SO(3) \times_{Z_6} D_1^{r+5}$ is $(S^1 + S^1) \times D^{k+1}$ and that on $SO(3) \times_{Z_6} D^{r+5}$ can be taken as $(S^1 + S^1) \times (D^{k+1} + (S^1 \times S^k))$. Thus, the fixed point set of Z_6 in λ is clearly $E^{k+2} + S^1 \times S^1 \times S^k + S^1 \times S^1 \times S^k$. Moreover, the representation α was altered only within a compact set and hence the action λ is linear at infinity and thus contains an action of $SO(3)$ on the disk D^{r+8} , linear on the boundary, such that the fixed point set of Z_6 is $D^{k+2} + S^1 \times S^1 \times S^k + S^1 \times S^1 \times S^k$.

REFERENCES

1. P. S. Mostert (Editor), *Proceedings of the conference on transformation groups*, Springer-Verlag, Berlin and New York, 1968.

RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY 08903