ON NORMAL COMPLEMENTS OF $\mathfrak{F}$ -COVERING SUBGROUPS

H. J. SCHMIDT, JR.

ABSTRACT. If $\mathfrak{F}$ is a suitably restricted formation, we show that an $\mathfrak{F}$ -covering subgroup H which is a Hall subgroup of the finite, solvable group G is complemented by the $\mathfrak{F}$ -residual of G , provided H normalizes an $\mathfrak{F}$ -normalizer of G . In particular, H is complemented by the $\mathfrak{F}$ -residual, if H is an $\mathfrak{F}$ -normalizer of G . Further, if $\mathfrak{F}$ is the class of nilpotent groups, then H complements the nilpotent residual, if G has pronormal system normalizers. Examples are given to show the necessity of the various hypotheses.

In this note all groups considered are finite and solvable. The notation and definitions are essentially those of [2]. Throughout we let $\mathfrak{F}$ be a formation which is locally induced by a class of nonempty, integrated formations $\mathfrak{F}(p)$, one for each prime p . If $\mathfrak{F}(p) = \{1\}$ for each prime p , then $\mathfrak{F}$ is the formation of nilpotent groups.

THEOREM A. *Let $\mathfrak{F}$ be a formation, and let G be a finite, solvable group with $\mathfrak{F}$ -covering subgroup H . If*

- (1) *H is a Hall subgroup of G , and*
- (2) *H normalizes an $\mathfrak{F}$ -normalizer of G , then H is complemented by the $\mathfrak{F}$ -residual N of G .*

PROOF. Because H covers G/N , it suffices to show that $H \cap N = 1$. Let G be a minimal counter-example, and let A be a minimal normal subgroup of G . Suppose that H is a π -Hall subgroup of G , where π is a set of primes.

Since G/A satisfies the hypotheses, it follows that $H \cap N \leq A$ and that NA/A is a π' -Hall subgroup of G/A . Therefore, A is the unique minimal normal subgroup of G . If A is a π' -group then N would be a π' -Hall subgroup of G , and $H \cap N = 1$, a contradiction. Thus, $A = H \cap N$ and A is a p -group for $p \in \pi$.

Let S be a p -complement of N . Using the Frattini argument, we may write $G = AM$, where $M = N(S)$. If $A \leq M$, then S would be normal in G . Therefore $A \cap M = 1$, $A = C(A)$, and M is a maximal subgroup of G .

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If A is $\mathfrak{F}$ -central, then $G/A \in \mathfrak{F}(p) \subseteq \mathfrak{F}$, so that $A = N$. A result of Carter and Hawkes [2, 5.15] then implies that $N \cap H = 1$, a contradiction. Thus A is $\mathfrak{F}$ -eccentric.

Let D be an $\mathfrak{F}$ -normalizer of G normalized by H . We may choose $D \leq H$, so that $A \leq N(D)$. Since $D \cap A = 1$, it follows that $[A, D] \leq A \cap D = 1$, hence $D \leq C(A) = A$. This contradiction completes the proof.

A subgroup H of the group G is called pronormal, if, for each $x \in G$, there is a $y \in \langle H, H^x \rangle$, such that $H^y = H^x$. When $\mathfrak{F}$ is the formation of nilpotent groups, the $\mathfrak{F}$ -normalizers of a group G are called system normalizers and the $\mathfrak{F}$ -covering subgroups are called Carter subgroups. Rose [3] has shown that the class of groups with pronormal system normalizers is a formation containing both the A -groups and the metanilpotent groups.

With this notion in mind we draw the following corollary to Theorem A.

COROLLARY. *Let G be a finite, solvable group with Carter subgroup H . If*

- (1) *H is a Hall subgroup of G , and*
- (2) *G has pronormal system normalizers,*

then H is complemented by the nilpotent residual N of G .

PROOF. Let D be a system normalizer of G contained in H . Since H is nilpotent D is subnormal in H . Therefore, D is both pronormal and subnormal in H , so [3, 1.4] implies that D is normal in H . Theorem A now applies.

Next we give examples to show that conditions (1) and (2) of Theorem A are necessary. To see that (2) is necessary, let G be the symmetric group on four letters. A 2-Sylow subgroup C of G is a Carter subgroup of G ; the alternating group on four letters is the nilpotent residual; and the 2-cycles are the system normalizers. The nilpotent residual does not complement C . Further, (12) is conjugate to (34) under (14)(23), but (14)(23) $\notin \langle (12), (34) \rangle$, so G does not have pronormal system normalizers.

To see that (1) is necessary, let G be the primitive, solvable group of degree 8 and order 168. The nilpotent residual is the elementary abelian group of order 8 extended by an automorphism of order 7. The Carter subgroup of G is the diagonal of the elementary abelian group extended by an automorphism of order 3. G has pronormal system normalizers since G is an A -group. However, the Carter subgroup of G is not a Hall subgroup of G and does not complement the nilpotent residual.

We conclude this note by generalizing the theorem in [4] from the formation of nilpotent groups to the arbitrary formations considered here.

THEOREM B. *Let $\mathfrak{F}$ be a formation, and let G be a finite, solvable group with $\mathfrak{F}$ -normalizer D . If D is a Hall subgroup of G , then D is complemented by the $\mathfrak{F}$ -residual N of G .*

PROOF. Because D covers G/N , it suffices to show that $D \cap N = 1$. Let G be a minimal counter-example, and let A be a minimal normal subgroup of G . Suppose that D is a π -Hall subgroup of G , where π is a set of primes.

Making a reduction analogous to that of Theorem A, we find that A is the unique minimal normal subgroup of G , $A \trianglelefteq D \cap N$, and $A = C(A)$. Since $A \leq D$, it follows that A is $\mathfrak{F}$ -central. Therefore, $G/A \in \mathfrak{F}(p) \subseteq \mathfrak{F}$. But then $A = N$, and again the result of Carter and Hawkes renders $N \cap D = 1$, the final contradiction.

A group G has property $P\mathfrak{F}$ if the $\mathfrak{F}$ -covering subgroups of G coincide with the $\mathfrak{F}$ -normalizers of G . By mimicking the proofs in [1] and [2], one can show that the class of groups with property $P\mathfrak{F}$ is a formation containing all extensions of a nilpotent group by a group belonging to $\mathfrak{F}$.

The above notion leads to the following corollary of Theorem B.

COROLLARY. *Let $\mathfrak{F}$ be a formation, and let G be a finite, solvable group with $\mathfrak{F}$ -covering subgroup H . If*

- (1) *H is a Hall subgroup of G , and*
- (2) *G has property $P\mathfrak{F}$,*

then H is complemented by the $\mathfrak{F}$ -residual N of G .

We comment that Sehgal and McWorter [4] have given examples to show the necessity of the hypotheses in the above corollary in case $\mathfrak{F}$ is the formation of nilpotent groups.

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ILLINOIS STATE UNIVERSITY, NORMAL, ILLINOIS 61761