IDENTITIES INVOLVING PRODUCTS OF NUMBER-THEORETIC FUNCTIONS

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ABSTRACT. General number-theoretic identities which involve various multiplications are given. The Tatzuza-Iseki identity which occurs in certain elementary proofs of the Prime Number Theorem is included as a special case.

For three different multiplications which are defined on the set of number-theoretic (arithmetical) functions we will use the notations

$$\begin{aligned} f \cdot g & \quad \text{if } (f \cdot g)(n) = f(n)g(n), \\ f * g & \quad \text{if } (f * g)(n) = \sum (f(k)g(m) : km = n), \\ f \# g & \quad \text{if } (f \# g)(x) = \sum (f(k)g(x/k) : 1 \leq k \leq x). \end{aligned}$$

It can be shown that the $\#$ -operation is not associative and not commutative, but the other two operations have both of these properties. Properties of the ring of number-theoretic functions with the $*$ -operation are discussed extensively in [1].

We shall use the following notations for functions: μ for the Möbius function, ν_0 for the constant function for which $\nu_0(n) = 1$, ϵ for the identity function with respect to the $*$ -operation, l for the logarithm function, and Λ for the von Mangoldt function for which $\Lambda(n) = \log p$ if $n = p^m$ and $\Lambda(n) = 0$ otherwise. In this notation we have $\mu * \nu_0 = \epsilon$ and $\mu * l = \Lambda$.

IDENTITY A. $f \# (g \# h) = (f * g) \# h$.

Direct rearrangement of the sums gives us

$$\begin{aligned} (f \# (g \# h))(x) &= \sum (f(k) \sum (g(m)h(x/km) : 1 \leq m \leq x/k) : 1 \leq k \leq x) \\ &= \sum (f(k)g(m)h(x/n) : km = n, 1 \leq n \leq x) \\ &= ((f * g) \# h)(x). \end{aligned}$$

The left inversion formulas for the $\#$ -operation can be obtained from Identity A; these follow from the Möbius inversion formulas for the $*$ -operation. In Identity A if we let $f = \nu_0$, $g = \mu$, $a = \mu \# h$, then $h = \nu_0 \# a$; if we let $f = \mu$, $g = \nu_0$, $b = \nu_0 \# h$, then $h = \mu \# b$. Two examples of other consequences are

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$$f \# (g \# h) = g \# (f \# h), \quad (a \# b) \# (c \# d) = c \# ((a \# b) \# d).$$

If in this last identity we set $c = \nu_0$, we can obtain

$$(a \# b) \# d = \mu \# ((a \# b) \# (\nu_0 \# d)).$$

IDENTITY B. $a \# (l \cdot (b \# c)) = (a * b) \# (c \cdot l) + (a * (b \cdot l)) \# c$.

If we use $l(x/k) = l(x/n) + l(n/k)$ along with a rearrangement of the sums, then

$$\begin{aligned} & (a \# (l \cdot (b \# c)))(x) \\ &= \sum (a(k)l(x/k)(b \# c)(x/k): 1 \leq k \leq x) \\ &= \sum (a(k)b(m)c(x/n)(l(x/n) + l(n/k)): mk = n, 1 \leq n \leq x) \\ &= \sum ((a * b)(n)(c \cdot l)(x/n): 1 \leq n \leq x) \\ &\quad + \sum ((a * (b \cdot l))(n)c(x/n): 1 \leq n \leq x) \\ &= ((a * b) \# (c \cdot l))(x) + ((a * (b \cdot l)) \# c)(x). \end{aligned}$$

An alternate form can be obtained by use of Identity A,

$$a \# (l \cdot (b \# c)) = a \# (b \# (c \cdot l)) + a \# ((b \cdot l) \# c).$$

One special case of interest is obtained by setting $a = \mu$ and $b = \nu_0$. Since ϵ is a left identity for the $\#$ -operation we have

$$\mu \# (l \cdot (\nu_0 \# c)) = c \cdot l + (\mu * (\nu_0 \cdot l)) \# c.$$

Since $\mu * (\nu_0 \cdot l) = \Lambda$, the Tatzuzawa-Iseki identity [2] which is used in some of the elementary proofs of the prime number theorem now follows;

$$\text{if } d = l \cdot (\nu_0 \# c), \quad \text{then } \mu \# d = c \cdot l + \Lambda \# c.$$

If we set $a = \nu_0$ and $b = \mu$ and use $\nu_0 * (\mu \cdot l) = -\Lambda$ we have analogously;

$$\text{if } d = l \cdot (\mu \# c), \quad \text{then } \nu_0 \# d = c \cdot l - \Lambda \# c.$$

For another example consider Liouville's function λ which is defined by $\lambda(n) = (-1)^{\Omega(n)}$ in which $\Omega(n)$ counts the total number of prime factors of n . Since $|\mu| * \lambda = \epsilon$ we have a number-theoretic identity analogous to the Tatzuzawa-Iseki identity;

$$\text{if } d = l \cdot (\lambda \# c), \quad \text{then } |\mu| \# d = c \cdot l + \Lambda' \# c$$

in which $\Lambda' = |\mu| * (\lambda \cdot l)$ so that

$$\begin{aligned} \Lambda'(n) &= \log p & \text{if } n &= p^{2m}, \\ &= -\log p & \text{if } n &= p^{2m+1}, \\ &= 0 & \text{otherwise.} \end{aligned}$$

Similarly we can obtain

$$\text{if } d = l \cdot (\mu \# c), \quad \text{then } \lambda \# d = c \cdot l - \Lambda' \# c.$$

IDENTITY B'. $a \# (l \cdot (b \# c)) = l \cdot ((a * b) \# c) - (b * (a \cdot l)) \# c.$

This is closely related to Identity B and it is derived in a similar manner, but by using $l(x/k) = l(x) - l(k)$. The Tatzuza-Iseki identity and its analogs are again special cases.

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