GRAPHS ASSOCIATED WITH A GROUP[1]

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ABSTRACT. It is shown that every group is isomorphic to a maximal group of graphs under the operation of graph product. In addition, maximal groups associated with an order graph are investigated.

By a *directed graph* X is meant a nonempty set $V=V(X)$ called the vertices of X, together with a set $E=E(X)$ called the edges of X consisting of ordered pairs of distinct elements from V. In general the notation follows [6].

For X a graph, $a\in V(X)$ and $A\subseteq V(X)$ let

$$aX=\{b\in V(X) \mid (a,b)\in E(X)\},\quad AX=\bigcup\{aX \mid a\in A\},\text{ and } XA=AX^*,$$

where X^* denotes the converse of the graph X.

The *product* XY of two graphs X and Y with common vertex set V is defined to be the graph Z where $aZ=(aX)Y$ for all $a\in V$. Geometrically this means that the row of Z for the vertex a consists of all vertices which can be reached from a by an edge sequence of length two in which the first edge belongs to X and the second to Y. Clearly the graph product operation is associative. Of particular interest will be the case where a collection $\mathcal{G}$ of graphs with vertex set V forms a group. By a *maximal group of graphs* on V is meant one that is not properly contained in any group of graphs with vertex set V.

The main result follows.

THEOREM 1. *Let G be any group. Then there exists a maximal group $\mathcal{G}$ of graphs with common vertex set, isomorphic to G.*

In order to prove the theorem, a special class of graphs will be introduced. A graph X is said to be *simple* provided that whenever $a,b\in V(X)$ and $A,B\subset V(X)$, aX nonempty and $aX=AX$ implies $a\in A$ and Xb nonempty and $Xb=XB$ implies $b\in B$. The group $\mathcal{G}$ in the theorem will be a collection of simple graphs.

Now a graph 0 is a [*partial*] *order* graph on a subset V' of V if $E(0)\subseteq V'\times V'$ and the relation $\leqq$ defined on V' by $a\leqq b$ if and only

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if $(a, b) \in E(0)$ constitutes a partial order relation on V'. The following lemma identifies this class of graphs.

LEMMA. *The class of simple idempotent graphs with vertex set V is the class of partial order graphs on subsets of V.*

PROOF. Let X be a simple idempotent. Then for $a \in XV$, $aX = (aX)X$ implies $a \in aX$ and for $b \in VX$, $Xb = X(Xb)$ implies $b \in Xb$, since X is simple. Thus $XV = VX$ and, letting $VX = V'$, $(a, a) \in E(X)$ for all $a \in V'$. Now if $(a, b) \in E(X)$ where $a \neq b$, then $aX \neq bX$ implies $(b, a) \notin E(X)$. Clearly since $X^2 = X$, (a, b), $(b, c) \in E(X)$ imply $(a, c) \in E(X)$ so X induces a reflexive, antisymmetric, transitive relation on V' and is thus a partial order graph.

If 0 constitutes a partial order graph then certainly $0^2 = 0$ and if $a0 = H0$ for some $a \in 0V$ then $a \in h0$ for some $h \in H$ so (h, a), $(a, h) \in E(0)$ imply $a = h \in H$. Similarly $b \in V0$ and $Vb = 0H$ imply $b \in H$. Thus 0 is simple.

PROOF OF THEOREM 1. Let G be a group. R. Frucht [3] has shown that there exists a partial order graph 0 such that G is isomorphic to $A(0)$, the automorphism group of 0. Let $V = 0V = V0$ be the vertex set for 0. By the lemma, 0 is a simple idempotent.

With each $\rho \in A(0)$ we associate a graph X with vertex set V, defined by $aX_\rho = (a\rho)0$ for each $a \in V$. Notice that $(a\delta)0 = (a0)\delta$ whenever $\delta \in A(0)$. Let $\mathcal{G}$ denote the collection of all such graphs. Then each member of $\mathcal{G}$ is simple since 0 is simple; and, since if $X_\rho, X_\sigma \in \mathcal{G}$

$$aX_\rho X_\sigma = (a\rho)0X_\sigma = [(a\rho)0]\sigma = [a(\rho\sigma)]0 = aX_{\rho\sigma},$$

the map $\rho \rightarrow X_\rho$ is an isomorphism of $A(0)$ onto $\mathcal{G}$. Thus G is isomorphic to the group $\mathcal{G}$ of simple graphs on V.

Moreover, suppose $\mathcal{G} \subseteq \mathcal{H}$ where $\mathcal{H}$ is a group of graphs with vertex set V. Then 0 is the identity of $\mathcal{H}$ so if $X \in \mathcal{H}$, $X0 = 0X = X$ and for some $Y \in \mathcal{H}$, $XY = YX = 0$. From $YX = 0$ and $X0 = X$ it follows that each set $a0$ is a union of sets bX and vice-versa. Similarly since $XY = 0$ and $0X = X$, each set $0a$ is a union of sets Xb and vice-versa. Thus X is simple and $X = \rho 0 = 0\sigma$ for some permutations ρ, σ of V, since 0 is simple and $V0 = 0V = V$. But $\rho = \sigma$ since 0 is a partial order graph and so $X \in \mathcal{G}$. Thus $\mathcal{H} = \mathcal{G}$ and $\mathcal{G}$ is a maximal group of graphs isomorphic to G.

Finally, we consider the special case where the graph 0 constitutes a total order on its vertex set V. In [1], [2] it was shown that the automorphism group $A(0)$ of an order graph 0 is abelian iff it is isomorphic to a direct product of subgroups of the reals under addition.

Moreover if 0 constitutes a well ordering of V, then $A(0)$ is trivial, that is, it consists of the identity permutation of V [**4**].

Now by the proof of Theorem **1**, the maximal group of graphs whose identity 0 is a partial order on its vertex set $V = V0$, is isomorphic to the automorphism group $A(0)$. Thus we have the following combined result.

THEOREM 2. *Let* 0 *be an order graph on* $V = V0$ *and let* $\mathcal{G}_0$ *be the associated maximal group of graphs. Then* $\mathcal{G}_0$ *is abelian iff it is isomorphic to a direct product of subgroups of the reals under addition. Moreover, if* 0 *constitutes a well ordering then* $\mathcal{G}_0$ *is trivial. In particular* $\mathcal{G}_0$ *is trivial whenever* V *is finite.*

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