NOT ALL SEMIREGULAR URYSOHN-CLOSED SPACES ARE KATĚTOV-URYSOHN

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ABSTRACT. A topological space is said to be *Urysohn* if every pair of distinct points have disjoint closed neighborhoods. In this note we give an example of a first countable semiregular Urysohn space which is closed in every Urysohn space in which it can be embedded, and on which there exists neither a coarser minimal Urysohn topology nor a coarser minimal first countable Urysohn topology.

For a topological property P , a P space is *minimal P* if there is no strictly coarser P topology and *P -closed* if it is closed in every P space containing it; a space is *Katětov- P* if there is a coarser minimal P topology. It is known [1], [2], [3], [6], [8], [9], [10] that a P -closed space is Katětov- P whenever $P =$ Hausdorff, first countable Hausdorff, completely Hausdorff, first countable regular, completely regular, first countable completely regular, normal, first countable normal, completely normal, first countable completely normal, perfectly normal, first countable perfectly normal, paracompact, first countable paracompact, or metric. In this note we show that for $P =$ Urysohn or first countable Urysohn, a P -closed space may *not* be Katětov- P ; this also answers a question posed in [4].

CONSTRUCTION OF THE SPACE (X, τ) . Let I_1, I_2, I_3 , and I_4 be pairwise disjoint, dense subsets of the unit interval I whose union is I , $I_5 = I_1$, and $X = \bigcup_{i=1}^5 I_i x \{i\}$. A set U is open precisely when

- (a) for $i = 1, 3$, or 5 , $(x, i) \in U$ if and only if there is an open set V in I (usual topology) such that $(x, i) \in Vx \{i\} \cap X \subseteq U$, and
- (b) for $i = 2$ or 4 , $(x, i) \in U$ if and only if there is an open set V in I such that $(x, i) \in Vx \{i-1, i, i+1\} \cap X \subseteq U$.

Clearly, (X, τ) is first countable, semiregular, and Urysohn.

For $\epsilon > 0$ and $x \in I$, let $S_\epsilon(x)$ denote the open interval $(x - \epsilon, x + \epsilon)$.

LEMMA. Let σ be a Urysohn topology on the set X such that $\sigma \subseteq \tau$.

- (a) If $a \leq b$ are real numbers, then the subspaces

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$$\bigcup_{i=1}^4 ([a, b] \cap I_i)x\{i\} \quad \text{and} \quad \bigcup_{i=2}^5 ([a, b] \cap I_i)x\{i\}$$

are Urysohn-closed.

(b) For $x \in I$ and $\epsilon > 0$, $\bigcup_{i=1}^5 (S_\epsilon(x) \cap I_i)x\{i\} \in \sigma$.

(c) For $i=1$ or 5 ,

$$\text{Cl}_\sigma((S_\epsilon(x) \cap I_i)x\{i\}) \subseteq \bigcup_{j=1}^5 ([x - \epsilon, x + \epsilon] \cap I_j)x\{j\}.$$

(d) Let $x \in I_i$ where $i=1$ or 5 . A set $A \subseteq X$ is a σ -neighborhood of (x, i) if and only if there is $\epsilon > 0$ such that $(S_\epsilon(x) \cap I_i)x\{i\} \subseteq A$.

PROOF. If for a space (Y, ρ) , ρ_* denotes the topology on Y which has $\{(\overline{T})^0 \mid T \in \rho\}$ as a base, then one can readily verify that (Y, ρ) is Urysohn-closed if and only if (Y, ρ_*) is Urysohn-closed. If (Y, ρ) is a space in (a), then one can show that (Y, ρ_*) is homeomorphic with a Urysohn-closed space that is produced by a construction in [5, Example 5]. (b) follows from (a), (c) follows from (b), and (d) follows from (a) and (b).

THEOREM. For $P = \text{Urysohn}$ or first countable Urysohn, (X, τ) is P -closed but not Katětov- P .

PROOF. By (a) of the lemma, it follows that (X, τ) is Urysohn-closed. Assume that (X, τ) is Katětov-Urysohn. Then there is a coarser minimal Urysohn topology σ on X . Let $x \in I_2$ and fix $i \in \{1, 5\}$. Using (c) of the lemma, it is easy to verify that the topology σ_x^i defined on X by $U \in \sigma_x^i$ precisely when $U \in \sigma$ and $(x, 2) \in U$ implies there is an $\epsilon > 0$ such that $(S_\epsilon(x) \cap I_i)x\{i\} \subseteq U$ is a coarser Urysohn topology; hence, $\sigma = \sigma_x^i$ for each x in I_2 . Let $y \in I_1$. There is $\epsilon > 0$ such that for $U_i = (S_\epsilon(y) \cap I_i)x\{i\}$ for $i=1$ and 5 , $\text{Cl}_\sigma(U_1) \cap \text{Cl}_\sigma(U_5) = \emptyset$. Since I_2 is dense in I , there is a point z in $S_\epsilon(y) \cap I_2$. Since $\sigma_z^i = \sigma$ for $i=1$ and 5 , then $(z, 2) \in \text{Cl}_\sigma((S_\epsilon(y) \cap I_i)x\{i\})$ for $i=1$ and 5 , a contradiction. This proves the theorem for the case $P = \text{Urysohn}$. The case $P = \text{first countable Urysohn}$ follows from the fact that (X, τ) is first countable and the proof of the case $P = \text{Urysohn}$.

A space is *completely Hausdorff* if and only if for every pair, x, y of distinct points, there exists a real-valued continuous function f such that $f(x) \neq f(y)$. As far as the author knows, the question for $P = \text{regular}$ and $P = \text{first countable completely Hausdorff}$ is unsolved.

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