A CHARACTERIZATION OF COVERING DIMENSION FOR COLLECTIONWISE NORMAL SPACES

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Ostrand [3] gave the following characterization of the covering dimension for metric spaces. A metric space X is of dimension $\leq n$ if and only if for each open cover C of X and each integer $k \geq n+1$ there exist k discrete families of open sets $U_1, U_2, \dots, U_k$ such that the union of any $n+1$ of the U_i is a cover of X which refines C . The purpose of this paper is to generalize Ostrand's theorem to collectionwise normal spaces by using an argument similar to that of Michael [1, Theorem 2].

DEFINITIONS. The collection H of subsets of the topological space X *strongly refines* the collection G of subsets of X if and only if for each $h \in H$ there is an element $g \in G$ such that $\bar{h} \subset g$, where $\bar{h}$ denotes the closure of h .

The collection H of subsets of the topological space X is σ_n -discrete if and only if H is the union of n discrete collections.

Suppose X is a topological space. The *covering dimension of X* is -1 , denoted $\dim X = -1$, if and only if X is empty. If n is a nonnegative integer, then the *covering dimension of X* is less than or equal to n , denoted $\dim X \leq n$ if and only if for each open cover G of X there is an open cover H of X such that H refines G and $\text{ord } H \leq n+1$.

REMARK. The usual definition for covering dimension is in terms of finite open covers. The above definition is equivalent to the usual definition of covering dimension for paracompact spaces. (See Theorem II.6, p. 22, of [2] and note that this theorem is true for any normal space and hence for any paracompact space.)

If X is a topological space, $M \subset X$, and G is a collection of subsets of X , then $\text{ord } {}_M G = \text{lub} \{ \text{ord } {}_p G : p \in M \}$ where $\text{ord } {}_p G$ is the number of elements of G containing p . Also, $\text{ord } {}_X G$ will be denoted $\text{ord } G$.

Notations. If H is a collection of sets, then πH denotes the common part of the elements of H and H^* denotes the union of the elements of H .

THEOREM 1. *If X is a collectionwise normal space, M is a closed subset of X , n is a positive integer, G is a collection each element of which is an open subset of X , and every point of M is in exactly n elements of G , then there is a discrete collection H of open subsets of X such that H covers M and strongly refines G .*

Received by the editors June 24, 1969.

PROOF OF THEOREM 1. Assume hypothesis. For each $p \in M$ let $R(p) = \{g \in G: p \in g\}$. If $p \in M$, then $R(p)$ has exactly n elements. Let $C = \{M \cap [\pi R(p)]: p \in M\}$.

It will now be shown that for all points $p, q \in M$ either $M \cap [\pi R(p)] = M \cap [\pi R(q)]$ or $(M \cap [\pi R(p)]) \cap (M \cap [\pi R(q)]) = \emptyset$ from which it follows that C is a discrete collection of closed point sets. Assume each of $p, q, t \in M$ and $t \in \{(M \cap [\pi R(p)]) \cap (M \cap [\pi R(q)])\}$. Each of $R(p)$ and $R(q)$ has exactly n elements. Each element of $R(p) \cup R(q)$ contains t , and t is contained by only n elements of G . Hence, $R(p) = R(q)$. Obviously $M \cap [\pi R(q)] = M \cap [\pi R(p)]$.

There is a discrete collection H of open sets such that each element of C is a subset of an element of H , no element of H intersects two elements of C , and H strongly refines G . H covers M since M is C^* .

THEOREM 2. *If X is a collectionwise normal space, $M \subset X$, M is closed, n is a nonnegative integer, G is a collection of open subsets of X covering M , and $\text{ord}_M G \leq n+1$, then there is a collection H of open subsets of X such that H covers M , H strongly refines G , and H is σ_{n+1} -discrete.*

PROOF OF THEOREM 2. By induction. The proposition is true for $n=0$ by Theorem 1. Assume that the proposition is true for the non-negative integer n . Assume the proposition hypothesis for the non-negative integer $n+1$. Let $R(p) = \{g \in G: p \in g\}$ for each $p \in M$. Let $A = \{p \in M: n+2 \text{ elements of } G \text{ contain } p\}$. If $p \in A$, then $M \cap [\pi R(p)]$ is open with respect to M and $M \cap [\pi R(p)] \subset A$. So $A = \{M \cap [\pi R(p)]: p \in A\}^*$ and hence A is open with respect to M . Now $M-A$ is closed and $\text{ord}_{M-A} G \leq n+1$. By the induction hypothesis there is a collection K of open subsets of X that covers $M-A$, strongly refines G , and is σ_{n+1} -discrete. Now $M-K^*$ is closed and $M-K^* \subset A$. So by Theorem 1 there is a discrete collection R of open subsets of X that covers $M-K^*$ and strongly refines G . $K \cup R$ is a collection of open subsets of X that covers M , strongly refines G , and is σ_{n+2} -discrete.

THEOREM 3. *The collectionwise normal space X has $\dim \leq n$ if and only if for each collection G of open sets covering X there is a collection H of open sets that covers X , strongly refines G , and is σ_{n+1} -discrete.*

PROOF OF THEOREM 3. The "if" part of the proof is obvious. The "only if" part of the proof will now be given. Suppose X is a collectionwise normal space and $\dim X \leq n$. Assume G is a collection of open sets covering X . There is a collection K of open sets such that K covers X , K refines G , and $\text{ord } K \leq n+1$. Now X is a closed subset of

X so by Theorem 2 there is a collection H of open sets such that H covers X , H strongly refines K , and H is σ_{n+1} -discrete.

THEOREM 4. *The collectionwise normal space S has $\dim \leq n$ if and only if for each open cover C of S and each integer $k \geq n+1$ there exist k discrete families of open sets $U(1), U(2), \dots, U(k)$ such that the union of any $n+1$ of the $U(i)$ is a cover of S which strongly refines C .*

PROOF OF THEOREM 4. The proof of the "if" part of the theorem is obvious. The proof of the "only if" part will now follow by induction. By Theorem 3, the proposition is true for $k = n+1$. Assume that the proposition is true for the integer $k \geq n+1$. There exist k discrete families of open sets $U(1), U(2), \dots, U(k)$ such that the union of any $n+1$ of the $U(i)$ is a cover of S which strongly refines C . Let $A = \{x: x \text{ is an } n\text{-termed subsequence of } 1, 2, \dots, k\}$. For each $x \in A$ let $H_x = S - [U(x(1)) \cup U(x(2)) \cup \dots \cup U(x(n))]^*$; hence, H_x is closed. Let $E = \{H_x: x \in A\}$. It will now be shown that if H_x and H_y are two nonempty elements of E , then H_x does not intersect H_y . Suppose H_x and H_y are two nonempty elements of E that intersect. Thus x is not y . There is a $p \in S - [U(x(1)) \cup \dots \cup U(x(n))]^*$ and $p \in S - [U(y(1)) \cup \dots \cup U(y(n))]^*$. There is a positive integer j such that $1 \leq j \leq n$ and $y(j)$ is not a term of x . Thus,

$$p \notin [U(y(j)) \cup U(x(1)) \cup U(x(2)) \cup \dots \cup U(x(n))]^*$$

but it is an open set collection covering S . Contradiction.

Now E is a finite collection of mutually exclusive closed point sets, so E is discrete. For each $x \in A$, let x' be a positive integer such that $1 \leq x' \leq k$ and x' is not a term of x . $U(x')$ must cover H_x . For each $x \in A$ let $E_x = \{h \cap H_x: h \in U(x')\}$ and note that E_x is a discrete refinement of $U(x')$ each element of which is closed. Let $T = \{E_x: x \in A\}^*$. Now T is a discrete refinement of $U(1) \cup U(2) \cup \dots \cup U(k)$ each element of which is closed. Hence T refines C . There is a discrete collection $U(k+1)$ of open sets that strongly refines C and covers T^* . Note that $T^* = E^*$. Now consider the sequence $U(1), \dots, U(k+1)$ and suppose that y is a subsequence of $1, 2, \dots, k+1$ with only $n+1$ terms. If $k+1$ is not a term of y , then by the induction hypothesis, $U(y(1)) \cup \dots \cup U(y(n+1))$ is a cover of S that strongly refines C . Suppose $k+1$ is a term of y . Let x be the n -termed sequence such that $x(i) = y(i)$ for $i = 1, 2, \dots, n$. Now $H_x \subset E^*$. Thus $U(k+1)$ covers H_x . So $U(y(1)) \cup \dots \cup U(y(n+1))$ covers S and strongly refines C .

COROLLARY 1. *The metric space X has $\text{Ind} \leq n$ if and only if for each open cover G of X and each integer $k \geq n+1$ there exist k discrete families*

of open sets $U(1), U(2), \dots, U(k)$ such that the union of any $n+1$ of the $U(i)$ is a cover of X which strongly refines G and if b is the boundary of an element of $U(i)$, then $\text{Ind } b \leq n-1$.

Ind denotes large inductive dimension. The proof of Corollary 1 follows easily from Theorem 4 and the fact the dim is equivalent to Ind in metric spaces.

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