UNITS IN MODULAR GROUP RINGS

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ABSTRACT. Let G be a finite p -group and let $U(G)$ denote the group of normalized units in the modular group ring of G . If G is nonabelian, then it is shown that the nonregular group $Z_p \wr Z_p$ is involved in $U(G)$. Here Z_p is the group with p elements and $Z_p \wr Z_p$ is wreath product.

Let $F[G]$ denote the group ring of a finite p -group G over the prime field $F = GF(p)$. We define the group of normalized units in $F[G]$ by

$$U(G) = \left\{ \sum_{x \in G} a_x \cdot x \in F[G] \mid \sum a_x = 1 \right\}.$$

We see easily that $|U(G)| = p^{|G|-1}$ and that $U(G)$ is in fact a normal Sylow p -subgroup of the full group of units in $F[G]$. If G is abelian, then $U(G)$ is abelian and its structure is therefore basically simple. On the other hand if G is nonabelian then the structure of $U(G)$ becomes immediately very complicated. As an indication of this fact we offer the following results.

THEOREM. *If G is nonabelian, then $Z_p \wr Z_p$ is involved in $U(G)$.*

COROLLARY. *If G is nonabelian, then $U(G)$ is not regular.*

The corollary follows immediately from the theorem since regularity is inherited by subgroups and quotient groups and since $Z_p \wr Z_p$ is known to be nonregular. We remark that the corollary is trivial for $p=2$ since a 2-group is regular if and only if it is abelian. On the other hand the proof of the theorem for $p=2$ requires the consideration of several special cases. For these, the following lemma is convenient. Here we let $o(x)$ denote the order of an element x .

LEMMA. *Let x and y be noncommuting elements of a 2-group P . Suppose that $o(x), o(y) \leq 4$ and furthermore that if $o(x) = o(y) = 4$ then $\langle x \rangle \cap \langle y \rangle = \langle 1 \rangle$. Then $Z_2 \wr Z_2$ is involved in P .*

PROOF. Clearly $Z_2 \wr Z_2$ is the dihedral group of order 8. Suppose first that $o(x) = o(y) = 2$. If $u = xy$ then we have easily $u^2 = u^{-1}$. Since x and u do not commute $o(u) \geq 4$ and thus if v is a suitable power of u

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with $o(v)=4$ then $\langle v, x \rangle \cong Z_2 \wr Z_2$. Now suppose that $o(x)=2$ and $o(y)=4$. Since x and y do not commute we have $x \neq y^2$. By the above we can assume that x and y^2 commute. If $\langle x, y \rangle = y^2$ then clearly $\langle x, y \rangle \cong Z_2 \wr Z_2$. On the other hand if $\langle x, y \rangle \neq y^2$ then the images of x and y in $\langle x, y \rangle / \langle y^2 \rangle$ are two noncommuting involutions so the result follows here. Finally let $o(x)=o(y)=4$ and $\langle x \rangle \cap \langle y \rangle = \langle 1 \rangle$. We can clearly assume that $(x, y^2) = (y, x^2) = 1$ so x^2 and y^2 are central in $\langle x, y \rangle$. Certainly (x, y) cannot equal both x^2 and y^2 so if $(x, y) \neq x^2$ then $\langle x, y \rangle / \langle x^2 \rangle$ contains two noncommuting elements one of order 2 and the other of order 4. The result follows.

We now proceed to prove the theorem. There are three cases to consider, the first two being of a special nature.

Case 1. Suppose that all subgroups of G are normal.

By Theorem 12.5.4 of [1] we have $p=2$ and $G=Q \times A$ where Q is the quaternion group of order 8 and A is elementary abelian. Let $Q=\langle x, y \rangle$ and consider x and $u=1+x+y$ as elements of $U(G)$. We see easily that $o(x)=o(u)=4$, $\langle x \rangle \cap \langle u \rangle = \langle 1 \rangle$ and $(x, u) \neq 1$. Thus by the lemma $Z_2 \wr Z_2$ is involved in $U(G)$.

Case 2. Suppose that $p=2$ and that G has a nonnormal subgroup of order 2.

Let $\langle x \rangle$ be the given nonnormal subgroup so $o(x)=2$ and x is not central in G . Let $y \in G$ with $(x, y) \neq 1$. If $o(y) \leq 4$ then by the lemma $Z_2 \wr Z_2$ is involved in G and hence in $U(G)$. Now let $o(y) \geq 8$. Then there are four nonidentity elements of $\langle y \rangle$ having the same fourth power as y . Thus we can choose $z \in \langle y \rangle$ with $z^4 = y^4$ and $z \neq 1, y, y^2$. Let x and $u=1+y+z$ be elements of $U(G)$. Since $u^x = 1+y^x+z^x$ we see by assumption that $(u, x) \neq 1$. Also since y and z commute and $y^4 = z^4$ we have $u^4 = 1$. Thus $o(x)=2$, $o(u) \leq 4$ so by the lemma $Z_2 \wr Z_2$ is involved in $U(G)$.

Case 3. Suppose finally that G has a nonnormal subgroup and that if $p=2$ then this subgroup has order ≥ 4 .

Let A be a nonnormal subgroup of G of maximal possible order and let B be a subgroup with $A < B \subseteq G$ and $|B:A| = p$. Then $B \triangleleft G$ and all conjugates of A are contained in B . Let $y \in G - N_G(A)$ with $y^p \in N_G(A)$ and set $A_i = A^{y^i}$ for $i=0, 1, \dots, p-1$. These are p distinct subgroups of B of index p . Hence $A_i \triangleleft B$.

Let $x_i = 1 + \hat{A}_i$ where $\hat{A}_i$ denotes the sum of the elements of A_i in $F[G]$. Now if $i \neq j$ then $A_i A_j = B$ and hence since $p \nmid |A_i|$ we have

$$\begin{aligned} (\hat{A}_i)^2 &= |A_i| A_i = 0, \\ (A_i)(A_j) &= |A_i \cap A_j| \hat{B}, \quad \text{for } i \neq j, \\ (A_i)(A_j)(A_k) &= 0, \quad \text{for all } i, j, k. \end{aligned}$$

Note that for $i \neq j$, $|A_i \cap A_j| = |B|/p^2 = |A|/p$. Let X be the subgroup of $U(G)$ generated by the x_i . We show that X is elementary abelian of order p^p .

By the above X is clearly abelian. Also $x_i^{a_i} = 1 + a_i \hat{A}_i$ so $x_i^p = 1$ and X is elementary abelian. Now if $0 \leq a_i < p$ then we have easily

$$\prod x_i^{a_i} = 1 + \sum a_i \hat{A}_i + b \hat{B}$$

where $b = (|A|/p) \sum_{i < j} a_i a_j$. Suppose that $\prod x_i^{a_i} = 1$. Since $B > \cup A_i$ by order considerations it follows that the coefficient of any element of $B - \cup A_i$ in $\prod x_i^{a_i}$ is equal to b and hence $b = 0$. Then since $A_i > \cup_{j \neq i} (A_j \cap A_i)$ we see that $a_i = 0$. This implies that $|X| = p^p$.

Now $x_i^y = x_{i+1}$ (subscripts modulo p) so $y \in U(G)$ normalizes X . Set $Y = \langle X, y \rangle$. We show that this is a split extension, that is, that $X \cap \langle y \rangle = \langle 1 \rangle$. Let $w \in X \cap \langle y \rangle$. Since $w \in \langle y \rangle$ we have $w \in G$. Clearly y centralizes w so we must have $w = \prod x_i^a$ for some a . Thus by the above

$$w = \prod x_i^a = 1 + \sum a A_i + b \hat{B}$$

where $b = (|A|/p)(p(p-1)/2)a^2$. If $p > 2$ then $p(p-1)/2$ is zero in F while if $p = 2$ then $|A|/p$ is zero in F by assumption. Hence in either case $b = 0$. The coefficient of the identity in the above right-hand expression is then $1 + pa = 1$ so since $w \in G$ we have $w = 1$.

Finally y^p is clearly central in Y and since $\langle y^p \rangle \cap X = \langle 1 \rangle$ we have $Y/\langle y^p \rangle \cong Z_p \wr Z_p$ is involved in $U(G)$. This completes the proof of the theorem.

REFERENCE

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