DIVERGENCE OF INTERPOLATION POLYNOMIALS IN THE COMPLEX DOMAIN¹

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ABSTRACT. In 1918 L. Fejér gave an example of a function $f(z)$, analytic for $|z| < 1$ and continuous for $|z| \leq 1$, such that the sequence of Lagrange polynomials found by interpolation to $f(z)$ at the roots of unity diverges at a point on the unit circle. More recently S. Ja. Al'per showed that, regardless of how the interpolation points are chosen on the unit circle, a function $g(z)$, analytic for $|z| < 1$ and continuous for $|z| \leq 1$, exists such that the Lagrange polynomials do not converge uniformly to $g(z)$ for $|z| \leq 1$. In the present paper we present some theory which sheds some light on the results of Fejér and Al'per. A new example of the divergence of Lagrange polynomials is also presented.

1. Introduction. Let Γ be an arbitrary Jordan curve in the complex z -plane. For each n , $n = 0, 1, \dots$ let $S_n = \{z_{n0}, z_{n1}, \dots, z_{nn}\}$ be a set of $n+1$ distinct points on Γ . If $f(z)$ is a complex-valued function defined on Γ , let $L_n(f; z)$ denote the polynomial of degree at most n which interpolates to $f(z)$ at the points of S_n . This polynomial is given by the familiar formula:

$$(1.1) \quad L_n(f; z) = \sum_{k=0}^n f(z_{nk}) \lambda_{nk}(z)$$

where

$$\begin{aligned} \lambda_{nk}(z) &= \omega_n(z) / \omega_n'(z_{nk})(z - z_{nk}), \\ \omega_n(z) &= (z - z_{n0})(z - z_{n1}) \cdots (z - z_{nn}). \end{aligned}$$

Let $A(\Gamma)$ denote the class of complex-valued functions which are analytic on $\text{Int } \Gamma$ and continuous on $\bar{\Gamma} = \Gamma \cup \text{Int } \Gamma$. For the case in which Γ is the unit circle, S_n is the set of $(n+1)$ st roots of unity, and $f \in A(\Gamma)$, Fejér [8] proved that the sequence $\langle L_n(f; z) \rangle$ converges to $f(z)$ uniformly on compact subsets of $|z| < 1$. This result has been extended to more general curves Γ by various authors; see for example Curtiss

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[5], [6], [7], Al'per [1], Thompson [15], and O'Hara [12]. With Γ the unit circle and S_n the roots of unity, Fejér in [8] also exhibited a function $f^* \in A(\Gamma)$ such that at a point z_0 on Γ , $\langle L_n(f^*; z_0) \rangle$ is unbounded. Also Al'per [1] showed that no matter how S_n is chosen on the unit circle Γ , there exists $g^* \in A(\Gamma)$ such that $L_n(g^*; z)$ does not converge uniformly to $g(z)$ for $|z| \leq 1$.

In this paper we propose to complement these divergence examples with some theory and further examples. In §2 necessary and sufficient conditions are given for the convergence, pointwise and uniform, of $\langle L_n(f; z) \rangle$ with $f \in A(\Gamma)$. In §3 these conditions are used to shed some light on the examples of Fejér and Al'per. In the last section, with Γ the unit circle, a specific family of functions in $A(\Gamma)$ and sets S_n are introduced for which $\langle L_n(f; z) \rangle$ is unbounded at a point of $\text{Int } \Gamma$.

2. Necessary and sufficient conditions for convergence of $\langle L_n(f; z) \rangle$.

Let Γ be an arbitrary Jordan curve and suppose $f \in A(\Gamma)$. By referring to the existence of convergent polynomial approximations for $f(z)$, z on Γ , Curtiss [5, p. 181] remarked that a sufficient condition that $\lim_{n \rightarrow \infty} L_n(f; z_0) = f(z_0)$ for any $z_0 \in \text{Int } \Gamma$ is that the sequence $\langle \lambda_n(z_0) \rangle_{n=0}^{\infty}$ shall be bounded, where $\lambda_n(z) = \sum_{k=0}^n |\lambda_{nk}(z)|$, $n=0, 1, \dots$. The remark remains valid for z_0 on Γ . (The convergence is uniform on $\bar{\Gamma}$ if $\langle \lambda_n(z) \rangle$ is uniformly bounded on $\bar{\Gamma}$.) He conjectured that the condition is necessary. We here establish the conjecture by showing that if this sequence is unbounded, then there exist functions f in $A(\Gamma)$ for which $\langle L_n(f; z_0) \rangle$ is unbounded.

Our starting point is the following simple consequence of a theorem of Rudin [13] (see also [3]).

LEMMA 2.1. *Let Γ be an arbitrary Jordan curve in the complex z -plane and let $z_1, \dots, z_m$ be m distinct points on Γ . Let $t_1, \dots, t_m$ be any m complex numbers (not necessarily distinct). Then there exists $g \in A(\Gamma)$ such that $g(z_j) = t_j$, $j=1, \dots, m$. Moreover if $|t_j| \leq 1$, $j=1, \dots, m$, then g can be chosen so that $|g(z)| \leq 1$ for $z \in \bar{\Gamma}$.*

We now note that for any fixed z_0 and for each $n=0, 1, \dots$ the mapping $L_n(\cdot; z_0): f \rightarrow L_n(f; z_0)$ is a linear functional on the space of bounded complex functions on Γ normed with the sup norm. The operator norm $\|L_n(\cdot; z_0)\|$ is clearly $\lambda_n(z_0)$. There remains a question, here crucial, as to whether this is still the value of $\|L_n(\cdot; z_0)\|$ when the domain of $L_n(\cdot; z_0)$ is restricted to the Banach space $A(\Gamma)$. The answer is affirmative.

LEMMA 2.2. *The norm $\|L_n(\cdot; z_0)\|$ with $L_n(\cdot; z_0)$ considered as a linear functional on $A(\Gamma)$ is $\lambda_n(z_0)$.*

Let $s(z) = |z|/z$, $z \neq 0$, $s(0) = 1$, and on the points z_{nk} , $k = 0, \dots, n$, define a function g_0 by $g_0(z_{nk}) = s[\lambda_{nk}(z_0)]$. Then $|g_0(z_{nk})| = 1$ and $L_n(g_0; z_0) = \lambda_n(z_0)$. Now by Lemma 2.1, there exists a function $g(z) \in A(\Gamma)$ such that $g(z_{nk}) = g_0(z_{nk})$, $k = 0, \dots, n$, and $\sup_{z \in \bar{\Gamma}} |g(z)| = 1$. Thus by definition of the operator norm, $\|L_n(\cdot; z_0)\| = \lambda_n(z_0)$ even when the domain of $L_n(\cdot; z_0)$ is restricted to $A(\Gamma)$.

THEOREM 2.1. *If the sequence $\langle \lambda_n(z_0) \rangle$ is unbounded then the sequence $\langle L_n(f; z_0) \rangle$ is unbounded for all f belonging to some dense G_δ set in $A(\Gamma)$.*

This follows from the Banach-Steinhaus theorem [14, p. 98] now that the sequence $\langle \lambda_n(z_0) \rangle$ has been identified with the sequence of operator norms $\langle \|L_n(\cdot; z_0)\| \rangle$ with the domain of $L_n(\cdot; z_0)$ restricted to $A(\Gamma)$.

Theorem 2.1 deals with the pointwise behavior of the sequence $\langle L_n(f; z) \rangle$. The following theorem is the natural counterpart of Theorem 2.1 for the uniform behavior of this sequence.

THEOREM 2.2. *If the sequence $\langle \lambda_n \rangle$ is unbounded, where $\lambda_n = \sup_{z \in \bar{\Gamma}} \lambda_n(z)$, then the sequence $\langle L_n(f; z) \rangle$ is not uniformly bounded on $\bar{\Gamma}$ for all f belonging to some dense G_δ set in $A(\Gamma)$.*

Since the proof is similar to that of Theorem 2.1, we present only a brief outline and let the reader supply the details. Consider the linear operator $L_n: A(\Gamma) \rightarrow A(\Gamma)$ defined by $L_n: f \rightarrow L_n(f; z)$. The norm of this operator is given by $\|L_n\| = \sup_{z \in \bar{\Gamma}} \lambda_n(z) = \lambda_n$ (the last equality follows from the theory of subharmonic functions). An application of the Banach-Steinhaus theorem completes the proof.

3. Remarks on the examples of Fejér and Al'per. In order to examine the example of Fejér, we begin with the following

LEMMA 3.1. *If Γ is the unit circle and S_n is the set of $(n+1)$ st roots of unity, then the sequence $\langle \lambda_n(z_0) \rangle$ is unbounded at each point z_0 on Γ with the exception of $z_0 = 1$.*

For in this case $\omega_{n-1}(z) = z^n - 1$, and hence

$$\lambda_{n-1}(z_0) = \frac{|z_0^n - 1|}{n} \sum_{k=0}^{n-1} \frac{1}{|z_0 - z_{nk}|}$$

where $z_{nk} = \exp(2\pi i k/n)$. If $z_0 \neq 1$ then there is a subsequence $\langle n_j \rangle$ of the positive integers and a positive number δ such that $|z_0^{n_j} - 1| \geq \delta$, $j = 1, 2, \dots$. Let $z_0 = \exp(2\pi i \lambda)$, $0 < \lambda < 1$, and let $[x]$ denote the largest integer $\leq x$. Now apply the identity $\exp(i\theta) - \exp(i\alpha) = 2i \sin((\theta - \alpha)/2) \exp(i(\theta + \alpha)/2)$, to obtain for j sufficiently large:

$$\begin{aligned}\lambda_{n_j-1}(z_0) &\geq \frac{\delta}{2n_j} \sum_{k=0}^{n_j-1} \frac{1}{|\sin(\pi\lambda - \pi k/n_j)|} \geq \frac{\delta}{2\pi n_j} \sum_{k=0}^{[n_j]} \frac{1}{\lambda - k/n_j} \\ &\geq \frac{\delta}{2\pi} \int_{\lambda - [n_j\lambda]/n_j}^{\lambda} \frac{1}{x} dx \geq \frac{\delta}{2\pi} \int_{1/n_j}^{\lambda} \frac{1}{x} dx = \frac{\delta}{2\pi} \log n_j \lambda.\end{aligned}$$

Fejér's divergence example can now be viewed as a specific realization of the following theorem:

THEOREM 3.1. *Let Γ be the unit circle and let S_n be the set of $(n+1)$ st roots of unity. Let T be any countable subset of $\Gamma - \{1\}$. Then there is a dense G_δ set in $A(\Gamma)$ such that whenever f belongs to this set, the sequence $\langle L_n(f; z) \rangle$ is unbounded at each $z \in T$.*

Let $T = \{z_1, z_2, \dots\}$ and let A_j be the set of $f \in A(\Gamma)$ such that $\sup_n L_n(f; z_j) = \infty$. Then by Theorem 2.1 and the lemma above, each A_j contains a dense G_δ set in $A(\Gamma)$. Therefore by Baire's category theorem $\bigcap_{j=1}^\infty A_j$ contains a dense G_δ set in $A(\Gamma)$.

As usual in polynomial approximation theory, relatively light further restrictions on functions in $A(\Gamma)$ may exclude them from the G_δ set of divergence referred to in Theorem 3.1. For example, with Γ and S_n as in the preceding theorem, it is proved in [7] that when $f \in A(\Gamma)$ is of bounded variation on Γ , then $\lim_{n \rightarrow \infty} L_n(f; z) = f(z)$ uniformly on $\bar{\Gamma}$. (See also [1, Theorem 2].) We also remark that the results of [4] can be used to extend Lemma 3.1 (and hence Theorem 3.1) to a more general type of Jordan curve Γ . In this extension the interpolation points S_n are the so-called Fejér points on Γ (see for example [7]).

We turn now to examine Al'per's result, for which purpose the following lemma is needed.

LEMMA 3.2. *For any choice of S_n , $n = 0, 1, \dots$, on the unit circle Γ , the sequence $\langle \lambda_n \rangle$ is unbounded.*

Al'per [1] mentions that this lemma can be proved by slightly modifying the proof of a similar theorem due to Faber and Bernštejn (see [11, p. 370]). We shall outline here a different proof. Choose $f \in A(\Gamma)$ and z_0 on Γ so that $T_n(z_0)$ diverges, where $T_n(z)$ is the n th partial sum of the Taylor series for $f(z)$ (see for example [17, p. 300]). Define $g_t \in A(\Gamma)$ by $g_t(z) = f(tz)$ and let $G_n(t) = L_n(g_t; z_0/t)$. Now suppose to the contrary that $\lambda_n \leq M$, $n = 0, 1, \dots$. Then for each t on Γ , $\langle \lambda_n(z_0/t) \rangle$ is bounded and by Curtiss' remark: $\lim_{n \rightarrow \infty} G_n(t) = g_t(z_0/t) = f(z_0)$. Moreover $\langle G_n(t) \rangle$ is uniformly bounded by $M(\sup_{z \in \Gamma} |f(z)|)$. Hence if $\langle \lambda_n \rangle$ is bounded then

$$(3.1) \quad \lim_{n \rightarrow \infty} \frac{1}{2\pi i} \int_{|t|=1} \frac{G_n(t)}{t} dt = \frac{1}{2\pi i} \int_{|t|=1} \frac{f(z_0)}{t} dt = f(z_0).$$

However this is impossible since it can be shown that for each $n=0, 1, \dots$ the integral in the left member of (3.1) equals $T_n(z_0)$. (A similar result for trigonometric interpolation is proved in [10].)

Applying Theorem 2.2 we can slightly improve Al'per's result as follows:

THEOREM 3.2. *For any choice of $S_n, n=0, 1, \dots$, on the unit circle Γ , there is a dense G_δ set in $A(\Gamma)$ such that whenever f belongs to this set, the sequence $\langle L_n(f; z) \rangle$ is not uniformly bounded on $\bar{\Gamma}$.*

An apparently open question in connection with the results of this section is as follows: If $\langle S_n \rangle$ and Γ are arbitrary, does there exist z_0 on Γ such that $\langle \lambda_n(z_0) \rangle$ is unbounded? (Bernšteĭn [2] has answered this question in the affirmative when Γ is replaced by an interval on the real line.) In fact it is apparently unknown whether or not Lemma 3.2 holds for an arbitrary Γ .

4. Example of divergence for z_0 in the interior of the unit disc. Let Γ be the unit circle. For β irrational, $0 < \beta < 1$, let $S_n^{(\beta)} = \{z_{n0}, \dots, z_{nn}\}$ where $z_{nk} = \exp(2\pi i k/n)$, $0 \leq k \leq n-1$, and $z_{nn} = \exp(2\pi i \beta)$. Also define a function $f_\beta \in A(\Gamma)$ by:

$$f_\beta(z) = (z - \exp(2\pi i \beta))^{1/2}$$

where

$$(w)^{1/2} = |w|^{1/2} \exp(\arg w/2), \quad 2\pi\beta \leq \arg w < 2\pi\beta + 2\pi.$$

The sequence $\langle S_n^{(\beta)} \rangle$ is uniformly distributed (see [16, p. 164]) on Γ . Therefore by a well-known theorem of Fejér [8]: $\lim_{n \rightarrow \infty} L_n(f; z) = f(z)$ uniformly on $\bar{\Gamma}$, providing $f(z)$ is analytic on $\bar{\Gamma}$. However $\sup_n \lambda_n(z_0) = \infty$ for every $z_0 \in \text{Int } \Gamma$ (note that $\lambda_{nn}(z_0) = (z_0^n - 1)/(\exp(2\pi i n\beta) - 1)$), and so for each $z_0 \in \text{Int } \Gamma$ there are functions $f \in A(\Gamma)$ for which $\langle L_n(f; z_0) \rangle$ is unbounded. The following theorem shows that in the case $z_0 = 0$, one of these functions is f_β (at least for some β 's).

THEOREM 4.1. *Let $L_n(f_\beta; z)$ be the Lagrange polynomial which interpolates to $f_\beta(z)$ at the points of $S_n^{(\beta)}$. Then there is a dense set of β 's for which $\langle L_n(f_\beta; 0) \rangle$ is unbounded.*

After some computation, it is found from (1.1) that, with square roots determined as in the definition of f_β :

$$L_n(f_\beta; 0) = -\frac{e^{i2\pi\beta}}{n} \sum_{k=0}^{n-1} (z_{nk} - e^{i2\pi\beta})^{-1/2}$$

$$= -\frac{e^{i2\pi\beta}}{(2)^{1/2}n} \sum_{k=0}^{n-1} \left(\sin\left(\frac{\pi k}{n} - \pi\beta\right) \exp\left(i\left(\frac{\pi k}{n} + \pi\beta + \frac{\pi}{2}\right)\right) \right)^{-1/2}.$$

Now let $[n\beta]$ denote the largest integer $\leq n\beta$. Also let $\psi_{nk} = \pi k/n + \pi\beta + 3\pi/2$ and $\phi_{nk} = \pi k/n + \pi\beta + \pi/2$. Then the above equation can be written:

$$L_n(f_\beta; 0) = -\frac{e^{i2\pi\beta}}{(2)^{1/2}n} \left[\sum_{k=0}^{[n\beta]} \left(\sin\left(\pi\beta - \frac{\pi k}{n}\right) \exp(i\psi_{nk}) \right)^{-1/2} \right. \\ \left. + \sum_{k=[n\beta]+1}^{n-1} \left(\sin\left(\frac{\pi k}{n} - \pi\beta\right) \exp(i\phi_{nk}) \right)^{-1/2} \right].$$

It follows from the definition of ψ_{nk} and ϕ_{nk} that:

$$(4.1) \quad \begin{aligned} \pi\beta + 3\pi/2 &\leq \psi_{nk} < 2\pi\beta + 3\pi/2; & 0 \leq k \leq [n\beta], \\ 2\pi\beta + \pi/2 &< \phi_{nk} < \pi\beta + 3\pi/2; & [n\beta] + 1 \leq k \leq n-1. \end{aligned}$$

Since $0 < \beta < 1$ we also obtain:

$$(4.2) \quad \begin{aligned} 2\pi\beta + \pi/2 &< \psi_{nk} < 2\pi\beta + 3\pi/2; & 0 \leq k \leq [n\beta], \\ 2\pi\beta + \pi/2 &< \phi_{nk} < 2\pi\beta + 3\pi/2; & [n\beta] + 1 \leq k \leq n-1. \end{aligned}$$

Now note that: $-\pi < \pi k/n - \pi\beta < 0$ if $0 \leq k \leq [n\beta]$, and $0 < \pi k/n - \pi\beta < \pi$ if $[n\beta] + 1 \leq k \leq n-1$. Therefore:

$$(4.3) \quad \begin{aligned} \sin(\pi\beta - \pi k/n) &> 0; & 0 \leq k \leq [n\beta], \\ \sin(\pi k/n - \pi\beta) &> 0; & [n\beta] + 1 \leq k \leq n-1. \end{aligned}$$

By applying (4.2), (4.3), and the definition of the square root used to define f_β , the last expression for $L_n(f_\beta; 0)$ can be rewritten (all square roots from here on will be nonnegative):

$$L_n(f_\beta; 0) = - (e^{i2\pi\beta}/2) [A_n^{(\beta)} - iB_n^{(\beta)}],$$

where

$$A_n^{(\beta)} = \sum_{k=0}^{[n\beta]} \frac{\cos(\psi_{nk}/2)}{n(\sin(\pi\beta - \pi k/n))^{1/2}} + \sum_{k=[n\beta]+1}^{n-1} \frac{\cos(\phi_{nk}/2)}{n(\sin(\pi k/n - \pi\beta))^{1/2}}$$

and

$$B_n^{(\beta)} = \sum_{k=0}^{[n\beta]} \frac{\sin(\psi_{nk}/2)}{n(\sin(\pi\beta - \pi k/n))^{1/2}} + \sum_{k=[n\beta]+1}^{n-1} \frac{\sin(\phi_{nk}/2)}{n(\sin(\pi k/n - \pi\beta))^{1/2}}.$$

We now consider the following cases individually: $0 < \beta < 1/4$, $1/4 < \beta < 1/2$, $1/2 < \beta < 3/4$, $3/4 < \beta < 1$. In each case we show that there is a dense set of β 's for which either $\langle A_n^{(\beta)} \rangle$ or $\langle B_n^{(\beta)} \rangle$ is unbounded. If $0 < \beta < 1/4$ then by (4.1): $3\pi/4 < \psi_{nk}/2 < \pi$ if $0 \leq k \leq [n\beta]$, and $\pi/4 < \phi_{nk}/2 < 7\pi/8$ if $[n\beta] + 1 \leq k \leq n-1$. Thus all the terms of $B_n^{(\beta)}$ are positive and so:

$$\begin{aligned} B_n^{(\beta)} &> \frac{\sin(\phi_{n, [n\beta]+1}/2)}{n(\sin((\pi([n\beta] + 1)/n) - \pi\beta))^{1/2}} \\ &> \frac{\sin(7\pi/8)}{\pi^{1/2}(n(1 - (n\beta - [n\beta])))^{1/2}}. \end{aligned}$$

Using standard results in the theory of continued fractions (see for example [9, Chapter X]) it can be shown there is a dense set of β 's such that both the sequences

$$\langle n(n\beta - [n\beta]) \rangle_{n=1}^{\infty} \text{ and } \langle n(1 - (n\beta - [n\beta])) \rangle_{n=1}^{\infty}$$

have subsequences which tend to zero. Hence $\langle L_n(f_\beta; 0) \rangle$ is unbounded for β belonging to a dense subset of $(0, 1/4)$. A similar analysis can be made in the remaining three cases.

REFERENCES

1. S. Ja. Al'per, *On the convergence of Lagrange's interpolational polynomials in the complex domain*, Uspehi Mat. Nauk **11** (1956), no. 5 (71), 44-50. (Russian) MR **18**, 728.
2. S. N. Bernšteĭn, *Sur la limitation des valeurs d'un polynôme $P_n(x)$ de degré n sur tout un segment par ses valeurs en $(n+1)$ points du segment*, Bull. Acad. Sci. URSS **1931**, no. 8, 1025-1050.
3. E. Bishop, *A general Rudin-Carleson theorem*, Proc. Amer. Math. Soc. **13** (1962), 140-143. MR **24** #A3293.
4. J. H. Curtiss, *Riemann sums and the fundamental polynomials of Lagrange interpolation*, Duke Math. J. **8** (1941), 525-532. MR **3**, 115.
5. ———, *Interpolation with harmonic and complex polynomials to boundary values*, J. Math. Mech. **9** (1960), 167-192. MR **22** #4890.
6. ———, *Polynomial interpolation in points equidistributed on the unit circle*, Pacific J. Math. **12** (1962), 863-877. MR **29** #403.
7. ———, *Convergence of complex Lagrange interpolation polynomials on the locus of the interpolation points*, Duke Math. J. **32** (1965), 187-204. MR **35** #1787.
8. L. Fejér, *Interpolation und konforme Abbildung*, Göttingen Nachr. **1918**, 319-331.
9. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Clarendon Press, Oxford, 1938; 4th ed., Oxford Univ. Press, London, 1960.
10. J. Marcinkiewicz, *Quelques remarques sur l'interpolation*, Acta Univ. Szeged **8** (1936/37), 127-130.
11. I. P. Natanson, *Constructive function theory*, GITTL, Moscow, 1949; German transl., Akademie-Verlag, Berlin, 1955; English transl., Ungar, New York, 1964. MR **11**, 591; MR **16**, 1100; MR **33** #4529a.

12. P. J. O'Hara, *A study of interpolation by complex polynomials*, Doctoral Dissertation, University of Miami, Coral Gables, Florida, 1968.
13. W. Rudin, *Boundary values of continuous analytic functions*, Proc. Amer. Math. Soc. 7 (1956), 808-811. MR 18, 472.
14. ———, *Real and complex analysis*, McGraw-Hill, New York, 1966. MR 35 #1420.
15. M. Thompson, *Complex polynomial interpolation to continuous boundary data*, Proc. Amer. Math. Soc. 20 (1969), 327-332.
16. J. L. Walsh, *Interpolation and approximation by rational functions in the complex domain*, 2nd ed., Amer. Math. Soc. Colloq. Publ., vol. 20, Amer. Math. Soc., Providence, R. I., 1956; 4th rev. ed., 1965. MR 36 #1672b.
17. A. Zygmund, *Trigonometric series*, Vol. I, 2nd rev. ed., Cambridge Univ. Press, New York, 1959. MR 21 #6498.

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