A GENERALIZATION OF ZARISKI'S MAIN THEOREM[1]

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ABSTRACT. This note extends techniques of Peskine and Zariski to prove a generalization of Zariski's Main Theorem which allows infinite integral extensions of finitely generated rings.

The purpose of this note is to extend Peskine's treatment of Zariski's Main Theorem to allow for infinite integral extensions or geometrically to the "factorization of an integral morphism by a morphism locally of finite type." The author assumes that the reader has a copy of [**3**] at hand for comparison in order to avoid needless repetition of detail.

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All rings are assumed to be commutative with a unit. R_s stands for R localized at the multiplicative set $S=\{1, s, s^2, s^3, \cdots\}$. Inclusions of one ring in another assume that the smaller ring has the same unit as the larger.

DEFINITION. Let $R\subset T$ be rings and P a prime ideal of T. Then P is isolated over $R\cap P$ if P is maximal and minimal with respect to the primes of T whose intersection with R is $R\cap P$.

THEOREM. *Let $R\subset T$ with R integrally closed in T such that there exist $t_1, \cdots, t_n\in T$ with T integral over $R[t_1, \cdots, t_n]$. If a prime ideal P of T is isolated over $P\cap R$, then there exists on $s\in R-P\cap R$ such that $T_s=R_s$.*

PROOF. The proof proceeds in three steps: (a) we reduce to the case $n=1$ following almost exactly Zariski's treatment [**4**, p. 523]; (b) we reduce the case $n=1$ to Peskine's Lemma 5; (c) we refer the reader to Peskine's Lemma 5 observing that Peskine's Lemma 3,

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proposition on p. 123, and Lemma 5 are only stated for the local case but his Lemma 3 is used in the global case. However only trivial adjustments of the proofs are needed to prove all three in the global case.

(a) First we need a lemma, which occurs in the middle of one of Peskine's proofs, which indicates where the two hypotheses of our version of the theorem are used.

LEMMA 1. *Let $R\subset T$ be rings such that there exists $t_i\in T$, $i=1,\cdots,n$, with T integral over $R[t_1,\cdots,t_n]$. Let P be a prime ideal of T maximal with respect to being disjoint from $M=R-P\cap R$. Let S be any subring of T which contains R. Then $P\cap S$ is maximal with respect to being disjoint from M.*

PROOF. Localizing R, S, and T on M we can assume that $P\cap R$ is maximal. Factoring out by $R\cap P$, $S\cap P$ and P we can assume that R and T are fields. Then $R[t_1,\cdots,t_n]$ is a field since T is integral over it and T is a field. But the only finitely generated algebras over fields that are fields are algebraic extensions. Hence $R[t_1,\cdots,t_n]$ is algebraic over R. Hence T is algebraic over R. But then S is a field since any subring of an algebraic extension is a field. Hence $P\cap S$ is maximal disjoint from M as desired.

Now assuming the theorem is true for $n=1$ we prove the general result by induction on n. Assume the theorem is true for m we want it for $m+1$. So we have the hypothesis with T integral over $R[t_1,\cdots,t_{m+1}]$. Let S be the integral closure of $R[t_1,\cdots,t_m]$. Then S is integrally closed in T, T is integral over $R[t_{m+1}]$ and P is isolated. Hence there exists an $r\in S-P\cap S$ such that $S_r=T_r$. Hence no prime of S contained in $P\cap S$ can intersect R in $P\cap R$. By Lemma 1 no prime containing $P\cap S$ can intersect R in $P\cap R$. Hence $P\cap S$ is isolated over $P\cap R$. By the case m of the theorem there exists an $r'\in R-P\cap R$ such that $R_{r'}=S_{r'}$. Then $r/1=r''/r'$ where $r''\notin P$ since r and $r'\notin P$. Let $s=r''r'$. Then $R_s=S_s$ and $S_s=T_s$. Hence $R_s=T_s$ as desired.

(b) First we need a lemma which in some form appears in Peskine [**3**, Lemma 2], Zariski [**4**, p. 524], and Krull [**2**, p. 135].

LEMMA 2. *Let $R\subset R[x]$ be rings with R integrally closed in $R[x]$, P a prime ideal of $R[x]$ and $Q=P\cap R$. Then either*

(1) *Every equation satisfied by x over R has all its coefficients in Q, or*

(2) *There exists an $s\in R-Q$ such that $R_s=R[x]_s$.*

PROOF. We assume that some equation satisfied by x over R has some coefficient in $R-Q$ and proceed by induction on the degree.

$n=1$. Then $rx-t=0$. If $r\notin Q$, then $x=t/r$ in R_r. Hence $R_r=R[x]_r$. If $r\in Q$, then $rx\in P$. Hence $t\in P\cap R=Q$.

General n. Say

$$(*)\qquad r_{n+1}x^{n+1} + \cdots + r_0 = 0$$

where the $r_i \in R$ and some $r_i \notin Q$. If $r_{n+1} \notin Q$, then x is integral over $R_{r_{n+1}}$ and x is in $R[x]_{r_{n+1}}$. Hence $R_{r_{n+1}} = R[x]_{r_{n+1}}$. If $r_{n+1} \in Q$, then multiplying (*) by r_{n+1}^n and regrouping shows that $r_{n+1}x$ is integral over R. Hence $r_{n+1}x \in R$. If $r_{n+1}x \notin Q$, then we are in the case $n=1$. If $r_{n+1}x \in Q$, then regrouping (*) as

$$(**)\qquad (r_{n+1}x + r_n)x^n + r_{n-1}x^{n-1} + \cdots + r_0 = 0$$

we get a new equation of degree n for x over R. If some r_i with $i<n$ was not in Q it still is not in Q. If $r_n \notin Q$, then $r_{n+1}x + r_n \notin Q$. Hence (**) is an equation of the desired type of lower degree. Hence there exists an $s \in R-Q$ such that $R_s = R[x]_s$ by the inductive hypothesis.

Now we reduce the $n=1$ case of the theorem to the case of $R \subset T$ with R integrally closed in T, T a finitely generated $R[t]$ module for some $t \in T$, and P a prime of T isolated over $P \cap R$. In the general $n=1$ case let $Q = P \cap R$. Then if (2) of Lemma 2 holds for $R \subset R[t]$ then $R_s = R[t]_s$ but R_s is integrally closed in T_s while T_s is integral over $R[t]_s$. Hence $R_s = T_s$. Thus (1) of the lemma holds.

Hence the kernel of the map from $R[X]$ to $R[t]$ given by sending X to t is contained in $QR[X]$. Thus the chains of prime ideals in $R[t]$ intersecting R in Q are all of length 1 with $QR[t]$ being the unique minimal one. If $P \cap R[t] = QR[t]$, then there would be a prime properly containing $P \cap R[t]$ also intersecting R in Q contrary to Lemma 1. Hence $P \cap R[t] \not\supseteq QR[t]$.

Pick $f \in P \cap R[t] - QR[t]$. P is minimal over QT. Hence there exists integers m and n greater than 0, an element $v \in T-P$, $q_i \in Q$, and $t_i \in T$ such that

$$(***)\qquad f^n v = \sum_{i=1}^{m} q_i t_i.$$

Let $T' = R[t, v, t_1, \cdots, t_m]$. Then T' is a finitely generated $R[t]$ module since T is integral over $R[t]$.

It is enough to show that $P \cap T'$ is isolated over Q. For then there would exist an $s \in R-Q$ such that $R_s = T'_s$ and R_s integrally closed in T_s while T_s is integral over T'_s. Thus $R_s = T_s$.

By the Lemma 1, no prime ideal of T' containing $P \cap T'$ properly can intersect R in Q. Hence, if $P \cap T'$ were not isolated, there would be a prime ideal P' of T' contained in $P \cap T'$ which intersected R in Q. (***) holds in T'. Hence P' must contain f. Hence $P' \cap R[t] \neq QR[t]$. Thus $P' \cap R[t] = P \cap R[t]$. But a chain of primes must have distinct intersection in any subring over which the ring is integral by Cohen and Seidenberg [**1**, Theorem 4]. Hence, $P' = P \cap R[t]$ is isolated and $T_s = R_s$ for some $s \in R - Q$.

(c) The final step is when T is a finitely generated $R[t]$ module. This is done in Peskine [**3**, pp. 123–125] using a conductor trick which was in Zariski's original paper [**4**, pp. 524–525].

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