REGULARITY OF BAIRE MEASURES

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ABSTRACT. In a recent paper N. Dinculeanu and I. Kluvánek showed that any Baire measure with values in a locally convex topological vector space is regular. Their construction depended heavily on the regularity of nonnegative Baire measures. In the present paper, a proof of the regularity is given which holds at once for the nonnegative case and the vector case.

I. The literature contains an abundance of proofs of the regularity of Baire measures. However, many of them, such as the one found in [**4**], require the proofs of a considerable number of preliminary lemmas before the actual theorem can be derived.

On the other hand, the proof of the regularity of vector-valued Baire measures found in [**1**] uses the regularity of positive Baire measures.

The following approach seems not only to require a minimum of machinery, but it also applies directly to the vector-valued measures considered in [**1**].

II. Let T denote a locally compact Hausdorff space, Δ the Baire σ-ring of subsets of T, X a locally convex topological vector space whose topology is given by a family $\{|\cdot|_p\}_{p\in P}$ of seminorms, and $m:\Delta\to X$ a (countably additive) Baire measure on T.

We say that m is *regular* at a set $A\in\Delta$ provided that for each neighborhood V of $0\in X$ there is a compact set $K\in\Delta$ and an open set $G\in\Delta$, with $K\subset A\subset G$, so that $m(S)\in V$, for each $S\in\Delta$, $S\subset G-K$. Such a pair (K, G) is said to be *appropriate* for (A, V). The measure m is said to be *regular* provided it is regular at each $A\in\Delta$.

For $p\in P$, we consider the p-quasi-variation $\tilde{m}_p$ defined for every $A\in\Delta$ by $\tilde{m}_p(A)=\sup\{|m(S)|_p:S\subset A,\ S\in\Delta\}$.

LEMMA. *The p-quasi-variation is positive, increasing, countably subadditive, and $\tilde{m}_p(A)<\infty$ for each $A\in\Delta$. Furthermore, $A_n\searrow\emptyset$ implies that $\tilde{m}_p(A_n)\searrow 0$, i.e. m is vsr (see* [**2**, III, 4.5] *and* [**3**, *Theorems* 2.6 *and* 2.7]).

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We are now able to prove the following theorem.

THEOREM. *Every Baire measure* $m:\Delta\to X$ *is regular.*

PROOF. Let $\mathcal{C}$ denote the class of sets of Δ for which m is regular. We shall prove that $\mathcal{C}$ coincides with Δ.

(a) $\mathcal{C}$ contains the compact G_δ sets. In fact, let K be a compact G_δ and let $\{G_n\}$ be a decreasing sequence of open Baire sets so that $G_n\searrow K$. Let $p\in P$, $\epsilon>0$, and $V=V(p,\epsilon)$ be the open $p-\epsilon$-ball around $0\in X$. Then $\tilde{m}_p(G_n-K)\searrow 0$, and there is an N so that $\tilde{m}_p(G_N-K)<\epsilon$. Hence if $S\in\Delta$ and S is contained in G_N-K, then $\tilde{m}_p(S)<\epsilon$. Thus $m(S)\in V$, and it follows that m is regular at K.

(b) $\mathcal{C}$ is a ring. Let A_1, $A_2\in\mathcal{C}$, and suppose that $V=V(p,\epsilon)$ is a neighborhood of $0\in X$. Let (K_i, G_i) be appropriate for $(A_i, V/2)$, $i=1, 2$. Then $(K_1\cup K_2, G_1\cup G_2)$ is appropriate for $(A_1\cup A_2, V)$. For if $S\in\Delta$ and S is a subset of $(G_1\cup G_2)-(K_1\cup K_2)$, then S can be written as $S_1\cup S_2$, $S_i\in\Delta$, $S_1\subset G_1-K_1$, and $S_2\subset G_2-K_2$. Therefore $m(S_1)+m(S_2)\in V$, and $A_1\cup A_2\in\mathcal{C}$.

In addition, it follows that (K_1-G_2, G_1-K_2) is appropriate for A_1-A_2. For

$$(G_1-K_2)-(K_1-G_2)=[G_1-(K_1\cup K_2)]\cup[(G_1\cap G_2)-K_2].$$

Therefore, if $S\in\Delta$ and $S\subset(G_1-K_2)-(K_1-G_2)$, then $S=S_1\cup S_2$, where $S_1\subset G_1-K_1$ and $S_2\subset G_2-K_2$. Thus $m(S_1)+m(S_2)\in V$, and it follows that $\mathcal{C}$ is a ring.

(c) $\mathcal{C}$ is a monotone class of sets. In fact, suppose that $\{A_n\}$ is a decreasing sequence of sets in $\mathcal{C}$, and let $A=\bigcap A_n$. Let $V=V(p,\epsilon)$ be a neighborhood of $0\in X$ as above, and for each n choose an appropriate pair for $(A_n, V/2^{2(n+1)})$. By taking intersections, we can generate a sequence $\{(K_n, G_n)\}$ so that, for each n, (K_n, G_n) is appropriate for $(A_n, V/2)$, and each of the sequences $\{K_n\}$ and $\{G_n\}$ is decreasing. The set $K=\bigcap K_n$ is compact, K belongs to Δ, and there is a positive integer N so that if $n\geqq N$, then $\tilde{m}_p(K_n-K)<\epsilon/2$. Now if S is a subset of G_N-K, then $S=S_1\cup S_2$, $S_i\in\Delta$, $S_1\subset G_N-K_N$, and $S_2\subset K_N-K$. Therefore $m(S)=m(S_1)+m(S_2)\in V$, and it follows that $A\in\mathcal{C}$.

By a dual argument, one can see that increasing sequences of elements of $\mathcal{C}$ also belong to $\mathcal{C}$. Hence $\mathcal{C}$ is a monotone ring containing all the compact G_δ sets, and therefore $\mathcal{C}=\Delta$ (see [4, Chapter 1]).

REMARK. As is proved in [1], every Baire measure $m:\Delta\to X$ can be uniquely extended to a regular Borel measure with values in the completion of X (see [1] for the definition of Baire and Borel sets).

REMARK. In order to simultaneously include the case of unbounded positive Baire measures (hence the values would not lie in a locally convex space), one may consider the following slightly more general setting. Let Δ_0 be the smallest ring and Δ_1 the smallest δ-ring generated by the compact G_δ subsets of T, let X be a locally convex space, and let $m:\Delta_1 \to X$ be a countably additive set function. As before, let $\mathcal{C}$ denote the class of all elements of Δ_1 at which m is regular. By the argument above, it follows that $\mathcal{C}$ is a ring containing the compact G_δ subsets of T and the intersection of a decreasing sequence of elements of $\mathcal{C}$ also belongs to $\mathcal{C}$. Furthermore, if $\{A_n\}$ is an increasing sequence of sets in $\mathcal{C}$ so that there is an $A \in \Delta_1$ so that $\{A_n\}_{n=1}^\infty \subset A$, then $\bigcup_{n=1}^\infty A_n \in \mathcal{C}$. Thus $\mathcal{C}$ is monotone with respect to Δ_1, and it follows that $\Delta_1 = \mathcal{C}$. To conclude, simply note that Δ is the collection of all countable unions from Δ_1.

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