LATTICE-INVARIANT PROPERTIES OF TOPOLOGICAL SPACES

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ABSTRACT. W. J. Thron proved in 1962 that regularity and normality are lattice-invariant properties but not T_0 and T_1 . In the present paper it is proved that complete regularity, compactness, local compactness, Lindelöf, second countability and connectedness are lattice-invariant properties. It is also proved that Hausdorff, complete normality, separability, and first countability are not lattice-invariant properties.

1. Two topological spaces are said to be *lattice-equivalent* if their lattices of open sets are isomorphic. It follows by duality that this is the case iff their lattices of closed sets are isomorphic. A topological property P is said to be *lattice-invariant* if a topological space lattice-equivalent to a topological space having P has also P .

W. J. Thron, in his paper [1], treated among others the lattice characterization of some separation axioms for topological spaces. He proved that T_0 and T_1 are not lattice-invariant. He also proved that regularity and normality are lattice-invariant.

In this paper, we shall prove that complete regularity, compactness, local compactness (in a sense), Lindelöf, second countability, and connectedness are lattice-invariant properties. We also prove that Hausdorff, complete normality, separability, and first countability are not lattice-invariant properties.

2. Let X be a topological space and $\mathcal{L}$ the lattice of all open sets in X . The following results are evident.

(i) X is compact iff for any family $\mathcal{D} \subset \mathcal{L}$ with $\bigvee \{D : D \in \mathcal{D}\} = X$, there is a finite subfamily $\mathcal{D}_0$ of $\mathcal{D}$ such that $\bigvee \{D : D \in \mathcal{D}_0\} = X$.

(ii) X is second countable iff there is a countable family $\mathcal{D} \subset \mathcal{L}$ such that any element in $\mathcal{L}$ is a lattice sum of some elements in $\mathcal{D}$.

(iii) X is Lindelöf iff for any family $\mathcal{D} \subset \mathcal{L}$ with $\bigvee \{D : D \in \mathcal{D}\} = X$, there is a countable subfamily $\mathcal{D}_0$ of $\mathcal{D}$ such that $\bigvee \{D : D \in \mathcal{D}_0\} = X$.

(iv) X is connected iff X is not a lattice sum of two disjoint non-zero elements in $\mathcal{L}$.

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Using the above lattice characterizations, we derive the following theorem.

THEOREM 1. *Compactness, second countability, Lindelöf and connectedness are lattice-invariant properties.*

3. Let X be a topological space. According to E. F. Steiner's definition, a family $\mathcal{C}$ of closed sets in X is said to be separating if for every closed set F in X and $x \in X \setminus F$, there exist $C_1, C_2 \in \mathcal{C}$ such that $F \subset C_1$, $x \in C_2$ and $C_1 \cap C_2 = \emptyset$. $\mathcal{C}$ is said to be normal if for every $A, B \in \mathcal{C}$ with $A \cap B = \emptyset$, there exist $C, D \in \mathcal{C}$ such that $C \cap B = \emptyset$, $A \cap D = \emptyset$ and $C \cup D = X$. In his paper [2], E. F. Steiner proved that a topological space is completely regular iff it possesses a normal separating family of closed sets.

Let $\mathfrak{L}$ be the lattice of all closed sets in X . Making use of the above statement, we shall prove that X is completely regular iff there is a subset $\mathcal{C}$ of $\mathfrak{L}$ such that the following two conditions are satisfied.

(i) For every $F \in \mathfrak{L}$ such that $F \neq X$, there exist sets $\{D_i : i \in I\} \subset \mathcal{C}$ and $\{E_i : i \in I\} \subset \mathcal{C}$ such that $F = \bigwedge \{D_i : i \in I\}$, $E_i \vee D_i = X$ and $E_i \wedge F = \emptyset$ for all $i \in I$.

(ii) For given $A, B \in \mathcal{C}$, with $A \cap B = \emptyset$, there exist $D, E \in \mathcal{C}$ such that $A \cap D = \emptyset$, $B \cap E = \emptyset$ and $D \vee E = X$.

PROOF. NECESSITY. Let X be a completely regular space and let $\mathcal{C}$ be a normal separating family of closed sets. We shall prove that $\mathcal{C}$ satisfies (i) and (ii).

(i) For $F \in \mathfrak{L}$, $F \neq X$ and any point $x \in X \setminus F$, there exist $A_x, B_x \in \mathcal{C}$ such that $A_x \cap B_x = \emptyset$ and $x \in A_x$, $B_x \supset F$. By the normality of $\mathcal{C}$, there are $E_x, D_x \in \mathcal{C}$ such that $E_x \cap B_x = \emptyset$, $D_x \cap A_x = \emptyset$ and $E_x \vee D_x = X$.

We shall prove that the sets $\{D_x : x \in X \setminus F\}$ and $\{E_x : x \in X \setminus F\}$ satisfy the requirements. Now $E_x \vee D_x = X$ for all $x \in X \setminus F$ follows from definition, and $E_x \cap F = \emptyset$ for all $x \in X \setminus F$ follows from $E_x \cap B_x = \emptyset$ and $B_x \supset F$. We now prove $F = \bigwedge \{D_x : x \in X \setminus F\}$. If $x \notin F$, then $x \in A_x$ and hence $x \notin D_x$. Therefore $x \notin \bigwedge \{D_x : x \in X \setminus F\}$. Conversely if $t \notin \bigwedge \{D_x : x \in X \setminus F\}$, then $t \notin D_x$ for some $x \notin F$. But $D_x \vee E_x = X$ and $E_x \cap B_x = \emptyset$, we get $t \notin B_x$ for some $x \notin F$. Now $B_x \supset F$ therefore $x \notin F$ and hence $F = \bigwedge \{D_x : x \in X \setminus F\}$.

(ii) The normality of $\mathcal{C}$ follows immediately.

SUFFICIENCY. Let $\mathcal{C}$ be the set of closed sets satisfying (i) and (ii).

Clearly, (ii) implies the normality of $\mathcal{C}$. To prove $\mathcal{C}$ is separating, let $x \in X \setminus F$ for some closed set F in X . By (i), F is the intersection of elements of $\mathcal{C}$, there we can find $C \in \mathcal{C}$ such that $x \notin C$ and $C \supset F$. Apply (i) again to the closed set $C \neq X$, we get $\{D_i : i \in I\} \subset \mathcal{C}$ and

$\{E_i: i \in I\} \subset \mathcal{C}$ satisfying requirement of (i). Now $x \notin C = \bigwedge \{D_i: i \in I\}$, therefore $x \notin D_i$ for some $i \in I$. It follows from $E_i \vee D_i = X$ that $x \in E_i$ for some $i \in I$. Therefore we get E_i and C of $\mathcal{C}$ such that $x \in E_i$, $C \supset F$ and $E_i \cap C = \emptyset$. $\mathcal{C}$ is then separating.

From this characterization, we easily obtain the following.

THEOREM 2. *Complete regularity is lattice-invariant.*

4. A topological space is said to be locally compact if every point in the space has a neighborhood with compact closure.

Let $\mathcal{L}$ be the lattice of all open sets in a topological space X . For $A \in \mathcal{L}$, let A' be the largest element in $\mathcal{L}$ such that $A \wedge A' = \emptyset$. A is said to be a PC -element if for $\mathcal{D} \subset \mathcal{L}$ such that $A' \vee (\bigvee \{D: D \in \mathcal{D}\}) = X$, there exists a finite subset $\mathcal{D}_0$ of $\mathcal{D}$ such that $A' \vee (\bigvee \{D: D \in \mathcal{D}_0\}) = X$. It is evident that X is locally compact iff $X = \bigvee \{D: D \in \mathcal{D}\}$ where $\mathcal{D}$ is the family of all PC -elements of $\mathcal{L}$.

From the above characterization, we have the following.

THEOREM 3. *Local compactness is lattice-invariant.*

5. Let X be a topological space consisting of only one point. X is lattice-equivalent to any indiscrete space. Therefore, Hausdorff is not a lattice-invariant property.

Let X be a discrete space containing uncountably many points. X is clearly completely normal and not separable. Let $Y = X \cup \{z\}$, where z is an element not in X . Let $\sigma = \{A \subset Y: z \in A\}$. Then σ and 2^X are isomorphic. But (Y, σ) is separable and is not normal hence not completely normal. This shows that complete normality and separability are not lattice-invariant properties.

Let X be the set of the ordinals less than the first uncountable ordinal number Ω . With the natural ordering, we define a topology for X in the following manner. Let M_x be the set consisting of all elements greater than x . The family τ consisting of X , $\emptyset$ and all M_x forms a topology for X . It is easy to verify that (X, τ) is first countable. Let $Y = X \cup \{\Omega\}$, with a topology σ defined similarly. It is clear that (Y, σ) is not first countable. But τ and σ are isomorphic. This shows that first countability is not lattice-invariant.

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