THE JACOBSON RADICAL OF THE ENDOMORPHISM RING OF A PROJECTIVE MODULE[1]

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ABSTRACT. In a recently published paper [3], the elements of the Jacobson radical of a ring of row-finite matrices over an arbitrary ring R are characterized as those matrices with entries in the Jacobson radical of R which have a vanishing set of column ideals. In this paper, the characterization is extended to include the endomorphism ring of an arbitrary projective module. In the process we offer a greatly simplified proof of the theorem for row-finite matrices.

Throughout R will be an associative ring, and Λ an infinite index set which we will assume to be well-ordered by the ordinals, $\Lambda = \{1, 2, \cdots\}$.

For a ring R, we will let $R^{\sharp}$ denote the ring obtained from R by adjoining an identity element in the customary manner, and $J(R)$ will denote the Jacobson radical of R.

Recall that for R a ring with 1, a left R-module P is *projective* if given any homomorphism $f: P \to N$ and any surjection $g: M \to N$ of left R-modules there exists a homomorphism $h: P \to M$ with $f = h \circ g$. It is well known [2, p. 86] that P is projective if and only if there exist elements $\{x_\lambda \mid \lambda \in \Lambda\} \subseteq P$ and homomorphisms $\{f_\lambda \mid \lambda \in \Lambda\} \subseteq \mathrm{Hom}_R(P, R)$ with $x = \sum_{\lambda \in \Lambda} (x f_\lambda) x_\lambda$ for each $x \in P$ (the sum having but finitely many nonzero terms). We will call such a family $\{(x_\lambda, f_\lambda) \mid \lambda \in \Lambda\}$ a *projective coordinate system* for P. We need one more definition before we can state our main theorem.

A set of left ideals $\{A_\lambda \mid \lambda \in \Lambda\}$ of a ring R is called a *vanishing set of left ideals* if given any sequence $a_1, a_2, \cdots$ with $a_i \in A_{\lambda_i}$ for distinct λ_i in Λ, there exists an integer n for which $a_1 a_2 \cdots a_n = 0$.

THEOREM 1. *Let P be a projective left R-module, R a ring with* 1, *and let $\phi \in E = \mathrm{Hom}_R(P, P)$, endomorphisms acting on the right. Then the following conditions are equivalent.*

(i) $\phi \in J(E)$.

(ii) *There exists an infinite projective coordinate system* $\{(x_\lambda, f_\lambda) \mid \lambda$

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$\in\Lambda\}$ *for* P *with* $\{P\phi f_\lambda | \lambda\in\Lambda\}$ *forming a vanishing set of left ideals contained in* $J(R)$.

(iii) *Given any infinite projective coordinate system* $\{(x_\lambda, f_\lambda) | \lambda\in\Lambda\}$ *for* P *with* $P\phi f_\lambda \subseteq J(R)$ *for each* λ, *the set* $\{P\phi f_\lambda | \lambda\in\Lambda\}$ *forms a vanishing set of left ideals.*

The hypothesis that the projective coordinate system be infinite is no real restriction, and is only made to include finitely generated projectives in the theorem. For if $\{(x_i, f_i) | i=1, 2, \cdots, n\}$ is a finite projective coordinate system for P, we can expand trivially to an infinite projective coordinate system by taking x_i arbitrary for $i>n$ and defining $f_i=0$ for $i>n$. The vanishing condition is then vacuously satisfied. We therefore have that $\phi\in J(E)$ if and only if for $i=1, \cdots, n$, $P\phi f_i \subseteq J(R)$. This is just a restatement of the fact that $J(E) = \mathrm{Hom}_R(P, J(P))$ for P a finitely generated projective.

Before we begin the proof of the theorem, we develop some lemmas which enable us to avoid some tedious matrix computations.

A submodule N of an R-module M is called *small* if whenever M_1 is a submodule with $N+M_1=M$, then already $M_1=M$. There is a characterization of the Jacobson radical of the endomorphism ring of a projective in terms of small submodules.

LEMMA 1. *Let* R *be a ring with* 1, P *a (quasi-) projective left* R*-module,* $E=\mathrm{Hom}_R(P, P)$. *Then* $J(E) = \{\phi\in E | P\phi$ *is a small submodule of* $P\}$.

PROOF. Set $T=\{\phi\in E | P\phi$ is a small submodule of $P\}$; it is clear that T is a left (in fact, two-sided) ideal of E. Given an arbitrary element $\phi\in T$, $P=P\phi+P(1-\phi)$. Since $P\phi$ is small in T, $P=P(1-\phi)$, so $1-\phi$ is a surjection. Since P is projective, there is a homomorphism $\psi: P\to P$ with $\psi(1-\phi)=1$. This shows that ϕ has a left quasi-inverse. Since ϕ was an arbitrary element of the left ideal T, $T\subseteq J(E)$.

For the reverse inclusion, let $\phi\in J(E)$ and suppose that $P\phi+Q=P$ for some submodule Q of P. Let π be the natural map of P onto P/Q. Since $P\phi+Q=P$, $P\phi\pi=P/Q$. Since P is projective, $\phi\pi$ lifts to a homomorphism $\psi: P\to P$ such that $\psi\phi\pi=\pi$. Thus $P(1-\psi\phi)\subseteq Q$. But $\phi\in J(E)$ implies that $1-\psi\phi$ is an automorphism of P, and it follows that $P=P(1-\psi\phi)=Q$. This proves that $J(E)\subseteq T$. Q.E.D.

LEMMA 2. *Let* R, P, E *be as in Lemma* 1. *Then* $J(E)\subseteq \mathrm{Hom}_R(P, J(P))$ *where* $J(P)$ *is the intersection of the maximal submodules of* P. *Consequently, given* $\phi\in J(E)$ *and* $f\in\mathrm{Hom}_R(P, R)$, $P\phi f\subseteq J(R)$.

PROOF. Suppose that $\phi\in J(E)$, and let M be any maximal submodule of P. Then either $P\phi\subseteq M$ or $M+P\phi=P$. The latter possibility cannot occur, because $P\phi$ is a small submodule of P by Lemma 1, which implies that $M=P$. M being an arbitrary maximal submodule of P, $P\phi\subseteq J(P)$ proving the first assertion.

The second assertion is an immediate consequence of the basic isomorphism theorems: $P\phi f\subseteq\bigcap$ (maximal left ideals of $R)=J(R)$ for any $f\in\mathrm{Hom}_R(P, R)$. Q.E.D.

It is known that $J(P)=J(R)P$; and that $J(E)=\mathrm{Hom}_R(P, J(P))$ for P finitely generated. But we will not need these facts here.

We will now prove the theorem by showing that (i) implies (iii), and is in turn implied by (ii).

(ii) *implies* (i). Suppose $\{(x_\lambda, f_\lambda)\mid\lambda\in\Lambda\}$ is a projective coordinate system for P, and with $\{P\phi f_\lambda\mid\lambda\in\Lambda\}$ a vanishing set of left ideals of R contained in $J(R)$. For convenience set $A_\lambda=P\phi f_\lambda$, $\lambda\in\Lambda$. For each $x\in P$, $x\phi=\sum_{\lambda\in\Lambda}(x\phi)f_\lambda x_\lambda\in\sum_{\lambda\in\Lambda}A_\lambda x_\lambda$. Thus $P\phi\subseteq\sum_{\lambda\in\Lambda}A_\lambda x_\lambda$.

We will be done if we can prove that $\sum_{\lambda\in\Lambda}A_\lambda x_\lambda$ is a small submodule of P; for then, a fortiori, $P\phi$ is a small submodule of P, and so by Lemma 1, $\phi\in J(E)$. Let us therefore assume that Q is a submodule of P with $\sum_{\lambda\in\Lambda}A_\lambda x_\lambda+Q=P$; our task is to prove that then $Q=P$.

Let x be an arbitrary element of P. Set $\bar{x}=x+Q\in P/Q$, $\bar{x}_\lambda=x_\lambda+Q\in P/Q$ for each $\lambda\in\Lambda$. We can write

$$\bar{x}=\sum_{\lambda_1\in\Lambda_1}c_{\lambda_1}\bar{x}_{\lambda_1} \tag{1}$$

where Λ_1 is a finite subset of Λ and $0\neq c_{\lambda_1}\in A_{\lambda_1}$ for each $\lambda_1\in\Lambda_1$. Next, for each $\lambda_1\in\Lambda_1$ we can write

$$\bar{x}_{\lambda_1}=\sum_{\lambda_2\in\Lambda_2}c_{\lambda_2}\bar{x}_{\lambda_2} \tag{2}$$

where Λ_2 is a finite subset of Λ (depending on the choice of λ_1) and $0\neq c_{\lambda_2}\in A_{\lambda_2}$ for each $\lambda_2\in\Lambda_2$. (The reader will note that we are avoiding some additional subscripting here. With this word of caution, no confusion should arise.)

We claim that we can assume that given $\lambda_1\in\Lambda_1$, $\lambda_1\notin\Lambda_2$. For one can use (2) to write $\bar{x}_{\lambda_1}=c_{\lambda_1}\bar{x}_{\lambda_1}+\sum_{\lambda_2\neq\lambda_1,\lambda_2\in\Lambda_2}c_{\lambda_2}\bar{x}_{\lambda_2}$ with $c_{\lambda_1}\in A_{\lambda_1}\subseteq J(R)$. Then $(1-c_{\lambda_1})\bar{x}_{\lambda_1}=\sum_{\lambda_2\neq\lambda_1,\lambda_2\in\Lambda_2}c_{\lambda_2}\bar{x}_{\lambda_2}$, so multiplying by $(1-c_{\lambda_1})^{-1}$,

$$\bar{x}_{\lambda_1}=\sum_{\lambda_2\in\Lambda_2;\,\lambda_2\neq\lambda_1}c'_{\lambda_2}\bar{x}_{\lambda_2},\ c'_{\lambda_2}\in A_{\lambda_2}. \tag{3}$$

Replacing equation (2) by equation (3) establishes the claim.

Thus $\bar{x}=\sum c_{\lambda_1}c_{\lambda_2}\bar{x}_{\lambda_2}$ with each nonzero pair c_{λ_1}, c_{λ_2} coming from distinct A_{λ_i}. Inductively, one can prove that for each integer $n\geqq 1$,

$\bar{x}=\sum c_{\lambda_1}c_{\lambda_2}\cdots c_{\lambda_n}\bar{x}_{\lambda_n}$ with each nonzero n-tuple of coefficients $c_{\lambda_1}, c_{\lambda_2}, \cdots, c_{\lambda_n}$ coming from distinct A_{λ_i}.

If $\bar{x}\neq 0$, then we have for each integer $n\geqq 1$ such a product $c_{\lambda_1}c_{\lambda_2}\cdots c_{\lambda_n}\neq 0$ (possibly distinct products for different n). But then by the Konig Graph Theorem, there would exist a sequence of elements $c_{\lambda_1}, c_{\lambda_2}, c_{\lambda_3}, \cdots$ with $\lambda_i\neq\lambda_j$ for $i\neq j$ and $c_{\lambda_1}c_{\lambda_2}\cdots c_{\lambda_n}\neq 0$ for every n. This would violate the hypothesis that $\{A_\lambda|\lambda\in\Lambda\}$ is a vanishing set of left ideals. Hence $\bar{x}=0$, and x being an arbitrary element of P it follows that $Q=P$, completing the proof that (ii) implies (i).

(i) *implies* (iii). Suppose $\phi\in J(E)$, and let $\{(x_\lambda, f_\lambda)|\lambda\in\Lambda\}$ be any projective coordinate system for P. For each $\lambda\in\Lambda$, we again set $A_\lambda=P\phi f_\lambda$. We note that by Lemma 2 each $A_\lambda\subseteq J(R)$. We have to prove that $\{A_\lambda|\lambda\in\Lambda\}$ is a vanishing set of left ideals of R.

Let a sequence $c_1\in A_{\lambda_1}, c_2\in A_{\lambda_2}, \cdots$ be given with the λ_i distinct elements of Λ. Without loss of generality we can assume that $\lambda_i=i$, $i=1, 2, 3, \cdots$. Write $c_i=p_i\phi f_i$, $i=1, 2, 3, \cdots$, where the $p_i\in P$.

We define a (free) R-module F as follows. Set $F=\sum_{\lambda\in\Lambda}\oplus R_\lambda$, where each $R_\lambda=R$; write the elements of F as $\sum_{\lambda\in\Lambda} r_\lambda e_\lambda$, where e_λ is the element of F with 1 in the λth coordinate, 0 elsewhere. There is an embedding ι of P into F defined by $x\iota=\sum_{\lambda\in\Lambda}(xf_\lambda)e_\lambda$. Moreover there is a homomorphism $\mu: F\to P$ defined by $(\sum_{\lambda\in\Lambda} r_\lambda e_\lambda)\mu=\sum_{\lambda\in\Lambda} r_\lambda x_\lambda$, with the property that $\iota\circ\mu$ is the identity map on P. It follows that $F=P\iota\oplus\ker\mu$. We can now extend ϕ to an endomorphism ϕ' of F: given $y\in F$, write $y=x\iota+z$ where $x\in P$ and $z\in\ker\mu$, and define $y\phi'=x\phi\iota$. Note that $F\phi'=P\phi\iota$. We also compute $(p_1\phi)\iota=c_1e_1+\sum_{\lambda\geq 2}(p_1\phi f_\lambda)e_\lambda$, and in general, $(p_i\phi)\iota=c_ie_i+\sum_{\lambda\neq i}(p_i\phi f_\lambda)e_\lambda$.

Choose a sequence $n_1, n_2, n_3, \cdots$ of positive integers as follows. Take $n_1=1$, and for $k\geqq 1$ and n_k having been chosen, inductively select $n_{k+1}>n_k$ so that $p_{n_k}\phi f_j=0$ for all integers $j\geqq n_{k+1}$. This selection is possible because the sum $\sum R_\lambda$ is direct.

Let $r_1, r_2, r_3, \cdots$ be an arbitrary sequence of elements of R. Define $\psi\in\mathrm{Hom}_R(F, F)$ via

$$\sum s_\lambda e_\lambda\in F, \quad (\sum s_\lambda e_\lambda)\psi=\sum t_\lambda e_\lambda,$$

where for each positive integer $j\geqq 2$, $t_j=s_{n_{j-1}}r_{j-1}$, and $t_\lambda=0$ otherwise. For any integer $k\geqq 1$, we compute

$$p_{n_k}\iota\phi'\psi=\sum_{i=1}^{k-1}(p_{n_k}\phi f_{n_i})r_ie_{i+1}+c_{n_k}r_ke_{k+1}=\sum_{i=1}^{k-1}s_{ki}r_ie_{i+1}+c_{n_k}r_ke_{k+1},$$

where $s_{ki}=p_{n_k}\phi f_{n_i}\in J(R)$.

Now since $P\phi\iota$ is a small submodule of $P\iota$, $F\phi' = P\phi\iota$ is a small submodule of F, and hence $\phi'\psi \in J(\mathrm{Hom}_R(F, F))$. It follows that $F\phi'\psi$ is a small submodule of F. Let $G = \sum_{k=1}^{\infty} R(e_k - c_{n_k} r_k e_{k+1}) + \sum_{\lambda \geq \omega} Re_\lambda$. We claim that $F\phi'\psi + G = F$. For, $e_1 = c_{n_1} r_1 e_2 + (e_1 - c_{n_1} r_1 e_2) = p_{n_1}\iota\phi'\psi + (e_1 - c_{n_1} r_1 e_2) \in F\phi'\psi + G$, so $e_1 \in F\phi'\psi + G$. Next, $(1 + s_{21} r_1) e_2 = (s_{21} r_1 e_2 + c_{n_2} r_2 e_3) + (e_2 - c_{n_2} r_2 e_3) = p_{n_2}\iota\phi'\psi + (e_2 - c_{n_2} r_2 e_3) \in F\phi'\psi + G$; and since $s_{21} r_1 \in J(R)$ it follows that $e_2 \in F\phi'\psi + G$. Inductively, one can perform a similar calculation to prove that $e_k \in F\phi'\psi + G$ for each positive integer k. Since by definition G contains all other e_λ we have $F\phi'\psi + G = F$. $F\phi'\psi$ being a small submodule of F, we conclude that $G = F$.

Thus we can write e_1 in terms of the generators of G,

$$e_1 = \sum_{i=1}^{k} a_i(e_i - c_{n_i} r_i e_{i+1}) + \sum_{\lambda \geq \omega} b_\lambda e_\lambda$$

with each a_i, $b_\lambda \in R$. Comparing coefficients, we see that each $b_\lambda = 0$, $a_1 = 1$, $a_2 = c_{n_1} r_1 = c_1 r_1$, $a_3 = a_2 c_{n_2} r_2 = c_1 r_1 c_{n_2} r_2, \cdots$, $a_k = a_{k-1} c_{n_{k-1}} r_{k-1} = c_1 r_1 c_{n_2} r_2 \cdots c_{n_{k-1}} r_{k-1}$, and finally $0 = a_k c_{n_k} r_k = c_1 r_1 c_{n_2} \cdots c_{n_k} r_k$. For $1 \leqq i \leqq k-1$, select $r_i = c_{n_i+1} \cdots c_{n_{i+1}-1}$ if $n_i + 1 < n_{i+1}$, and $r_i = 1$ otherwise; and choose $r_k = 1$. Then $c_1 c_2 c_3 \cdots c_{n_k} = 0$, and this completes the proof of the theorem.

Row-finite matrices. Given a ring R, we let R_f denote the ring of $\Lambda \times \Lambda$ row-finite matrices over R; $R_f^\#$ will denote the ring of $\Lambda \times \Lambda$ row-finite matrices over $R^\#$. It is convenient to regard R_f as a ring of endomorphisms, acting on the right, of a free left $R^\#$-module P with basis $\{x_\lambda \mid \lambda \in \Lambda\}$. This is possible by identifying $R_f \subseteq R_f^\# = \mathrm{Hom}_{R^\#}(P, P)$; and for $A = (a_{\mu\nu})_{\mu,\nu \in \Lambda} \in R_f$, and any $\mu \in \Lambda$, $x_\mu A = \sum_{\nu \in \Lambda} a_{\mu\nu} x_\nu$, the sum having but finitely many nonzero terms. We define the λth *column left ideal* of A to be the left ideal of R generated by $\{a_{\mu\lambda} \mid \mu \in \Lambda\}$. The following theorem was proved in [3], and is now an immediate corollary of Theorem 1.

THEOREM 2. *In order that $A = (a_{\mu\nu})_{\mu,\nu \in \Lambda} \in R_f$ be an element of $J(R_f)$, it is necessary and sufficient that each $a_{\mu\nu} \in J(R)$ and the column left ideals of A be a vanishing set of left ideals.*

PROOF. First assume that $R = R^\#$, and regard A as an element of $\mathrm{Hom}_R(P, P)$. For each $\lambda \in \Lambda$, define $f_\lambda \in \mathrm{Hom}_R(P, R)$ to be the natural projection homomorphism defined by $(\sum_{\lambda \in \Lambda} r_\lambda x_\lambda) f_\lambda = r_\lambda$. Then of course $\{(x_\lambda, f_\lambda) \mid \lambda \in \Lambda\}$ is a projective coordinate system for P.

Hence by Theorem 1, $A \in J(R_f)$ if and only if $\{PAf_\lambda \mid \lambda \in \Lambda\}$ is a vanishing set of left ideals contained in $J(R)$. But

$$PAf_\lambda = \left(\sum_{\mu \in \Lambda} Rx_\mu \right) Af_\lambda = \left(\sum_{\mu, \nu \in \Lambda} Ra_{\mu\nu} x_\nu \right) f_\lambda = \sum_{\mu \in \Lambda} Ra_{\mu\lambda},$$

which is just the λth column left ideal of A. This proves the theorem for rings with an identity element. For arbitrary rings the proof is completed by the following observation.

LEMMA 3. $J(R_f) = J(R_f^\#)$.

PROOF. We can apply Theorem 2 to $R^\#$ to learn in particular that $J(R_f^\#) \subseteq J(R^\#)_f$. (This is also easy to show directly.) So $J(R_f^\#) \subseteq J(R^\#)_f = J(R)_f \subseteq R_f$. Also by viewing R_f as a two-sided ideal of $R_f^\#$, we have $J(R_f) = R_f \cap J(R_f^\#)$. But $J(R_f^\#) \subseteq R_f$, so $J(R_f) = R_f \cap J(R_f^\#) = J(R_f^\#)$. Q.E.D.

Some concluding remarks are in order. The proof that (ii) implies (i) in Theorem 1 used a construction due to Bass [**1**, pp. 473–474]. The proof of the converse, while straightforward, is unsatisfactory in that one is forced to go outside the projective module to an associated free module. This should not be necessary.

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