ON $\mathfrak{F}$ -NORMALIZERS

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ABSTRACT. The concept of system normalizers of a finite solvable group is generalized to any saturated formation $\mathfrak{F}$. Given a Sylow system Σ of the finite solvable group G and an $\mathfrak{F}$ -projector V of G reducing Σ , a sequence $\{H_i\}$ of subgroups H_i , is constructed such that

- (i) each H_i reduces Σ , and
- (ii) the $\mathfrak{F}$ -normalizers of $\Sigma \cap H_i$ in H_i converge to V .

The concept of system normalizers has been extended by Carter and Hawkes [4] to any local formation $\mathfrak{F} = \{\mathfrak{F}(p)\}$, where $\mathfrak{F}(p) \subseteq \mathfrak{F}$ is a nonempty formation for each prime p . Using an idea of Fischer [5, 62–68], we give here an equivalent definition of $\mathfrak{F}$ -normalizers which is valid for all integrated (i.e., $\mathfrak{F}(p) \subseteq \mathfrak{F}$) local formation. Then we construct a sequence of $\mathfrak{F}$ -normalizers which converges to an $\mathfrak{F}$ -projector. The $\mathfrak{F}$ -projector is a name suggested by Gaschütz for what he originally called an $\mathfrak{F}$ -covering subgroup [6].

All groups considered are finite solvable. The notations and terminologies are standard. Given a group G and a formation $\mathfrak{F}$, $F(G)$ and $\Phi(G)$ denote respectively the Fitting subgroup and the Frattini subgroup of G , and $G(\mathfrak{F})$ is the least normal subgroup of G such that $G/G(\mathfrak{F}) \in \mathfrak{F}$. If a subgroup M of G reduces a Sylow system Σ of G , then $\Sigma \cap M$ denotes the Sylow system of M obtained by intersecting M with the members of Σ and $N_G(\Sigma \cap M)$ is the relative system normalizer. $\mathfrak{N}$ denotes the formation of all nilpotent groups, and $\mathfrak{N}\mathfrak{F}$ is the formation of groups which are extensions of nilpotent groups by groups in $\mathfrak{F}$.

1. $\mathfrak{F}$ -normalizers. Let $\mathfrak{F} = \{\mathfrak{F}(p)\}$ be an integrated local formation.

DEFINITIONS [4]. A p -chief factor H/K of a group G is $\mathfrak{F}$ -central of $G/C_G(H/K) \in \mathfrak{F}(p)$; otherwise, it is $\mathfrak{F}$ -eccentric. A maximal subgroup M of G is $\mathfrak{F}$ -normal if $M/\text{Core}(M) \in \mathfrak{F}(p)$, where p is the prime dividing $[G:M]$; otherwise, M is $\mathfrak{F}$ -abnormal.

We need the following results from [4] and [8]:

(A) [4, Lemma 2.3]. A maximal subgroup of G is $\mathfrak{F}$ -normal if and only if it complements an $\mathfrak{F}$ -central chief factor of G .

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(B) [4, Lemma 4.4]. $G \in \mathfrak{F}$ if and only if every minimal normal subgroup of $G/\Phi(G)$ is $\mathfrak{F}$ -central.

(C) [4, Theorem 5.8]. Let $G \in \mathfrak{N}\mathfrak{F}$ and H be a subgroup of G such that $H \in \mathfrak{F}$ and $G = F(G) \cdot H$. Then H is contained in some $\mathfrak{F}$ -projector of G .

(D) [8, Satz 2.1]. Let V be an $\mathfrak{F}$ -projector of a group G . Then $NM \cap V = (N \cap V)(M \cap V)$, for any normal subgroups N, M of G .

LEMMA 1.1. (a) $G(\mathfrak{F})$ covers all $\mathfrak{F}$ -eccentric chief factors of a group G .

(b) $M(\mathfrak{F}) \subseteq G(\mathfrak{F})$ for any $\mathfrak{F}$ -abnormal subgroup M of G .

PROOF. (a) It is sufficient to show that $G(\mathfrak{F})$ covers the $\mathfrak{F}$ -eccentric minimal normal subgroups. Let A be an $\mathfrak{F}$ -eccentric minimal normal p -subgroup, and let V be an $\mathfrak{F}$ -projector of the group G . Then

$$\begin{aligned} A &= A \cap A \cdot G(\mathfrak{F}) \cap V \cdot G(\mathfrak{F}) \\ &= A \cap [A \cdot G(\mathfrak{F}) \cap V] \cdot G(\mathfrak{F}) \\ &= A \cap [(A \cap V)(G(\mathfrak{F}) \cap V)] \cdot G(\mathfrak{F}) \quad (\text{by (D)}) \\ &= (A \cap V)(A \cap G(\mathfrak{F})). \end{aligned}$$

If $A \cap G(\mathfrak{F}) = 1$, then A centralizes $G(\mathfrak{F})$ and is a minimal normal p -subgroup of V . It follows that $G/C_G(A) \cong V/C_V(A) \in \mathfrak{F}(p)$. This is a contradiction to A being $\mathfrak{F}$ -eccentric.

(b) Let $K = \text{Core}(M)$ and let H/K be the chief factor complementing M . Then $H = (H \cap G(\mathfrak{F})) \cdot K$, $G = (H \cap G(\mathfrak{F})) \cdot M$, and $G(\mathfrak{F}) = (H \cap G(\mathfrak{F})) \cdot M(\mathfrak{F})$.

LEMMA 1.2. Let H be a normal subgroup of a group G . Then $N_G(\Sigma \cap H)$ reduces Σ , for any Sylow system Σ of G [7, Theorem 4.1].

PROOF. We have $G = H \cdot N_G(\Sigma \cap H)$. Let Σ_0 be a Sylow system of $N_G(\Sigma \cap H)$. Then $(\Sigma \cap H)\Sigma_0 = \{(S_p \cap H)T_p \mid S_p \in \Sigma, T_p \in \Sigma_0\}$ is a Sylow system of G . Hence $(\Sigma \cap H)\Sigma_0 = \Sigma^x$, for some $x \in G$. Since $\Sigma \cap H = (\Sigma \cap H)\Sigma_0 \cap H = \Sigma^x \cap H = (\Sigma \cap H)^x$, $x \in N_G(\Sigma \cap H)$. Therefore Σ reduces into $N_G(\Sigma \cap H)$.

Notations. $\mathfrak{D}(\Sigma)$ denotes the set of $\mathfrak{F}$ -projectors of $N_G(\Sigma \cap G(\mathfrak{F}))$ which reduce Σ . If H reduces Σ then $\mathfrak{D}(\Sigma \cap H)$ is the set of $\mathfrak{F}$ -projectors of $N_H(\Sigma \cap H(\mathfrak{F}))$ which reduce Σ .

Since $N_G(\Sigma \cap G(\mathfrak{F}))$ reduces Σ , $\mathfrak{D}(\Sigma)$ is not empty. The aim of this section is to show that $\mathfrak{D}(\Sigma)$ consists of only one element $D(\Sigma)$. Its proof, through a series of lemmas, follows closely the development in [5].

LEMMA 1.3. (a) If $G \in \mathfrak{N}\mathfrak{F}$ then $G = N_G(\Sigma \cap G(\mathfrak{F}))$, for any Sylow system Σ of G . Therefore the elements of $\mathfrak{D}(\Sigma)$ are $\mathfrak{F}$ -projectors of G .

(b) For any epimorphism α of G , $\mathfrak{D}(\Sigma)^\alpha \subseteq \mathfrak{D}(\Sigma^\alpha)$.

PROOF. Since $G(\mathfrak{F})$ is a nilpotent normal subgroup, (a) is valid. (b) also holds, because each step in the construction of $\mathfrak{D}(\Sigma)$ is invariant under epimorphisms.

LEMMA 1.4. Let Σ be a Sylow system of a group G , and let H be a normal subgroup containing $G(\mathfrak{F})$. Then the following statements hold:

(a) $N_G(\Sigma \cap H) \in \mathfrak{N}\mathfrak{F}$. Therefore the elements of $\mathfrak{D}(\Sigma \cap N_G(\Sigma \cap H))$ are $\mathfrak{F}$ -projectors of $N_G(\Sigma \cap H)$.

(b) Each element of $\mathfrak{D}(\Sigma \cap N_G(\Sigma \cap H))$ is contained in an element of $\mathfrak{D}(\Sigma)$.

PROOF. (a) Since $G = H \cdot N_G(\Sigma \cap H)$ and $H \supseteq G(\mathfrak{F})$,

$$N_G(\Sigma \cap H)/N_H(\Sigma \cap H) \cong G/H$$

is an epimorphic image of $G/G(\mathfrak{F})$. Therefore $N_G(\Sigma \cap H) \in \mathfrak{N}\mathfrak{F}$, and the elements of $\mathfrak{D}(\Sigma \cap N_G(\Sigma \cap H))$ are $\mathfrak{F}$ -projectors of $N_G(\Sigma \cap H)$, as shown in Lemma 1.3 (a).

(b) We use induction on the order $|G|$. Let G be a minimal counter-example and let H be of minimal index $[H:G(\mathfrak{F})]$ for which the lemma fails. Let H/K be a chief factor of $G/G(\mathfrak{F})$. Then each element of $\mathfrak{D}(\Sigma \cap N_G(\Sigma \cap K))$ is contained in an element of $\mathfrak{D}(\Sigma)$.

From

$$G = K \cdot N_G(\Sigma \cap K) = H \cdot N_G(\Sigma \cap K),$$

we have $N_G(\Sigma \cap K)/N_H(\Sigma \cap K) \cong G/H \in \mathfrak{F}$. Therefore the pair $N_G(\Sigma \cap K)$ and $N_H(\Sigma \cap K)$ satisfies the hypothesis of the lemma. Since $H = K \cdot N_H(\Sigma \cap K)$ and $N_G(\Sigma \cap H)$ normalizes $N_H(\Sigma \cap K)$, the relative system normalizer of $\Sigma \cap N_H(\Sigma \cap K)$ in $N_G(\Sigma \cap K)$ is $N_G(\Sigma \cap H)$. If $N_G(\Sigma \cap K) \neq G$ then, by induction, each element of $\mathfrak{D}(\Sigma \cap N_G(\Sigma \cap H))$ is contained in an element of $\mathfrak{D}(\Sigma \cap N_G(\Sigma \cap K))$. Hence each element of $\mathfrak{D}(\Sigma \cap N_G(\Sigma \cap H))$ is contained in an element of $\mathfrak{D}(\Sigma)$.

If $N_G(\Sigma \cap K) = G$ then K is nilpotent. Thus $G \in \mathfrak{N}\mathfrak{F}$. Let p be the prime dividing $|H/K|$. Then $H = (S_p \cap H)K$, where S_p is the Sylow p -subgroup in Σ . Then $N_G(\Sigma \cap H) = N_G(S_p \cap H) \supseteq S_p \cap H$. Hence

$$G = H \cdot N_G(\Sigma \cap H) = K \cdot N_G(\Sigma \cap H) = KE,$$

where

$$E \in \mathfrak{D}(\Sigma \cap N_G(\Sigma \cap H)).$$

By (C), E is contained in an $\mathfrak{F}$ -projector D of G . Since K is a nilpotent

normal subgroup, $D \cap K$ reduces any Sylow system of G . Therefore $D = (D \cap K)E$ reduces Σ , by [2, Lemma 2.7]. This completes the proof.

LEMMA 1.5. *Let M be an $\mathfrak{F}$ -abnormal subgroup of G which reduces a Sylow system Σ . Then each element of $\mathfrak{D}(\Sigma)$ is contained in an element of $\mathfrak{D}(\Sigma \cap M)$.*

PROOF. Let G be a minimal counterexample.

(1) $\mathfrak{D}(\Sigma) \subset M$. If $\text{Core}(M) \neq 1$ then $G/\text{Core}(M)$ satisfies the lemma. Therefore $\mathfrak{D}(\Sigma) \subset M$ if $\text{Core}(M) \neq 1$.

Suppose $\text{Core}(M) = 1$. Then there is a self-centralizing minimal normal p -subgroup A which complements M . We have $A \subseteq G(\mathfrak{F})$ and $M(\mathfrak{F}) = M \cap G(\mathfrak{F})$, by (A) and Lemma 1.1. If $M(\mathfrak{F}) = 1$ then M is an $\mathfrak{F}$ -projector of $G = N_G(\Sigma \cap G(\mathfrak{F}))$ [9]. Suppose that Σ also reduces into M^a , where $1 \neq a \in A$. Then the Sylow p -complement S^p of Σ is contained in M and M^a . Hence $[S^p, a] \subset M^a \cap A = 1$, and a centralizes $O_{p'}(M)$. This contradicts the assumption that A is self-centralizing. Hence M is the unique $\mathfrak{F}$ -projector reducing Σ . Suppose $M(\mathfrak{F}) \neq 1$. We show that $N_G(\Sigma \cap G(\mathfrak{F})) \subseteq M$. Suppose $am \in N_G(\Sigma \cap G(\mathfrak{F}))$, where $1 \neq a \in A$ and $m \in M$. Since $M \supseteq S^p$ and $S^p \cap G(\mathfrak{F}) = (S^p \cap G(\mathfrak{F}))^{am}$, we have $[S^p \cap G(\mathfrak{F}), a] \subseteq A \cap M = 1$. Then $a \in C_A(O_{p'}(M(\mathfrak{F})))$. Hence $A = C_A(O_{p'}(M(\mathfrak{F})))$. This contradicts the assumption that $\text{Core}(M) = 1$.

(2) Each element of $\mathfrak{D}(\Sigma)$ is contained in an element of $\mathfrak{D}(\Sigma \cap N_M(\Sigma \cap G(\mathfrak{F}) \cap M))$. Let $D \in \mathfrak{D}(\Sigma)$. We have

$$\begin{aligned} D &\subseteq N_G(\Sigma \cap G(\mathfrak{F})) \cap M = N_M(\Sigma \cap G(\mathfrak{F})) \\ &\subseteq N_M(\Sigma \cap G(\mathfrak{F}) \cap M), \\ G &= G(\mathfrak{F}) \cdot D, \end{aligned}$$

and

$$\begin{aligned} N_M(\Sigma \cap G(\mathfrak{F}) \cap M) &= [N_M(\Sigma \cap G(\mathfrak{F}) \cap M) \cap G(\mathfrak{F})] \cdot D \\ &= N_{G(\mathfrak{F}) \cap M}(\Sigma \cap G(\mathfrak{F}) \cap M) \cdot D. \end{aligned}$$

By (C), D is contained in an $\mathfrak{F}$ -projector E of $N_M(\Sigma \cap G(\mathfrak{F}) \cap M)$. Then $E = [E \cap N_{G(\mathfrak{F}) \cap M}(\Sigma \cap G(\mathfrak{F}) \cap M)] \cdot D$ reduces Σ , by [2, Lemma 2.7]. Therefore $E \in \mathfrak{D}(\Sigma \cap N_M(\Sigma \cap G(\mathfrak{F}) \cap M))$.

(3) Each element of $\mathfrak{D}(\Sigma \cap N_M(\Sigma \cap G(\mathfrak{F}) \cap M))$ is contained in an element of $\mathfrak{D}(\Sigma \cap M)$. The pair M and $G(\mathfrak{F}) \cap M$ satisfies the hypothesis of Lemma 1.4 (b). Therefore (3) is valid. This completes the proof.

THEOREM 1.1. $\mathfrak{D}(\Sigma)$ contains only one element.

PROOF. We use induction on $|G|$. Let G be a minimal counterexample. Then $G \notin \mathfrak{F}$ and there exists an $\mathfrak{F}$ -abnormal subgroup M , which reduces Σ and complements a p -chief factor $A/\Phi(G)$. Let P be the Sylow p -subgroup of $A \cap G(\mathfrak{F})$. Since A is nilpotent, P is a normal subgroup supplementing M . Since $G(\mathfrak{F}) = P \cdot M(\mathfrak{F})$, $N_M(\Sigma \cap M(\mathfrak{F})) \subseteq N_M(\Sigma \cap G(\mathfrak{F})) \subseteq N_G(\Sigma \cap G(\mathfrak{F}))$. By Lemma 1.5, each element of $\mathfrak{D}(\Sigma)$ is contained in an element of $\mathfrak{D}(\Sigma \cap M)$. Since $N_M(\Sigma \cap M(\mathfrak{F})) \subseteq N_G(\Sigma \cap G(\mathfrak{F}))$, each element of $\mathfrak{D}(\Sigma)$ is an element of $\mathfrak{D}(\Sigma \cap M)$. By induction, $\mathfrak{D}(\Sigma \cap M)$ has only one element. Therefore $\mathfrak{D}(\Sigma)$ has only one element.

DEFINITION. The unique element $D(\Sigma) = D$ of $\mathfrak{D}(\Sigma)$ is called the $\mathfrak{F}$ -normalizer of the Sylow system Σ .

We may restate Lemma 1.5 and the proof of Theorem 1.1 as follows:

THEOREM 1.2. *Let M be an $\mathfrak{F}$ -abnormal subgroup reducing a Sylow system Σ of G . Then $D(\Sigma \cap M) \supseteq D(\Sigma)$. If, in addition, M complements an $\mathfrak{F}$ -eccentric minimal normal subgroup of $G/\Phi(G)$, then $D(\Sigma \cap M) = D(\Sigma)$.*

REMARK. If $\mathfrak{F}$ contains the formation $\mathfrak{N}$ of nilpotent groups, then our definition of $\mathfrak{F}$ -normalizer coincides with the $\mathfrak{F}$ -normalizer of Carter and Hawkes.

PROOF. Let C_p be the intersection of all $C_G(H/K)$, where H/K runs over all $\mathfrak{F}$ -central p -chief factors. Put $D_0(\Sigma) = \bigcap_p N_G(S^p \cap C_p)$, where S^p is the Sylow p -complement in the Sylow system Σ . $D_0(\Sigma)$ is the $\mathfrak{F}$ -normalizer of Carter and Hawkes. If a maximal subgroup M reduces Σ and complements an $\mathfrak{F}$ -eccentric minimal normal subgroup of $G/\Phi(G)$, then [4, Lemma 4.6] $D_0(\Sigma) = D_0(\Sigma \cap M)$. Hence, by induction and Theorem 1.2, $D(\Sigma) = D(\Sigma \cap M) = D_0(\Sigma \cap M) = D_0(\Sigma)$.

The following theorem is [5, Theorem 8.5]. Fischer's proof, which we reproduce here, does not require $\mathfrak{F} \supseteq \mathfrak{N}$.

THEOREM 1.3. $D = D(\Sigma)$ covers all $\mathfrak{F}$ -central chief factors and avoids all $\mathfrak{F}$ -eccentric chief factors of G .

PROOF. By Lemma 1.3 (b), it is sufficient to show that D covers all $\mathfrak{F}$ -central minimal normal subgroups of G and avoids all $\mathfrak{F}$ -eccentric ones. Let A be an $\mathfrak{F}$ -central minimal normal p -subgroup of G . Since $G/C_G(A) \in \mathfrak{F}(p) \subseteq \mathfrak{F}$, A centralizes $G(\mathfrak{F})$. Consequently $A \subseteq N_G(\Sigma \cap G(\mathfrak{F}))$, and A is a minimal normal p -subgroup of AD . As $AD/C_{AD}(A) \cong G/C_G(A) \in \mathfrak{F}(p)$, we have $AD \in \mathfrak{F}$. Since D is an $\mathfrak{F}$ -projector of $N_G(\Sigma \cap G(\mathfrak{F}))$, $AD = D$.

Let A be an $\mathfrak{F}$ -eccentric minimal normal p -subgroup of G . We have $A \subseteq G(\mathfrak{F})$ and $A \cap D \subseteq G(\mathfrak{F}) \cap N_G(\Sigma \cap G(\mathfrak{F})) = N_{G(\mathfrak{F})}(\Sigma \cap G(\mathfrak{F}))$. Suppose $A \cap D \neq 1$. As a system normalizer covers the central chief factors and avoids the eccentric ones, $A \cap D \subseteq Z(G(\mathfrak{F}))$. Then $A \cap D$ is a normal subgroup. Consequently $A = A \cap D$ is also a minimal normal p -subgroup of D . Then $G/C_G(A) \cong D/C_D(A) \in \mathfrak{F}(p)$. This contradicts the assumption that A is $\mathfrak{F}$ -eccentric.

REMARK. A normal subgroup N of a group G is called $\mathfrak{F}$ -hypercentral [8], if all chief factors below N are $\mathfrak{F}$ -central. The $\mathfrak{F}$ -hypercenter of G is the maximal $\mathfrak{F}$ -hypercentral subgroup of G . Theorem 1.3 yields the usual characterization of $\mathfrak{F}$ -hypercenter as the intersection of all $\mathfrak{F}$ -normalizers.

2. A convergent sequence.

LEMMA 2.1. *Let Σ be a Sylow system of a group G , and $D = D(\Sigma)$. Then the following statements hold:*

- (a) *If an $\mathfrak{F}$ -projector V of G reduces Σ , then $V \supseteq D$.*
- (b) *$D(\Sigma \cap G(\mathfrak{N}\mathfrak{F}) \cdot D) \supseteq D$.*

PROOF. (a) Suppose there exists a maximal subgroup M which contains V and reduces Σ . By [4, Lemma 5.1], M is $\mathfrak{F}$ -abnormal. Then $D(\Sigma \cap M) \supseteq D$ by Theorem 1.2, and $V \supseteq D(\Sigma \cap M)$ by induction. It remains to produce such a maximal subgroup. Let A be a minimal normal subgroup of G . If $G = AV$, then V is maximal. If $G \neq AV$ then, by induction G/A has a maximal subgroup M/A which reduces $\Sigma A/A$ and contains AV/A . Then M reduces Σ and contains V .

(b) Since $G/G(\mathfrak{N}\mathfrak{F}) \in \mathfrak{N}\mathfrak{F}$, $G(\mathfrak{N}\mathfrak{F}) \cdot D/G(\mathfrak{N}\mathfrak{F})$ is an $\mathfrak{F}$ -projector of $G/G(\mathfrak{N}\mathfrak{F})$ by Lemma 1.3. From the above paragraph, there exists an $\mathfrak{F}$ -abnormal subgroup M_1 , which reduces Σ and contains $G(\mathfrak{N}\mathfrak{F}) \cdot D$. Repeat this process, we get a chain of subgroups $G = M_0 \supset M_1 \supset \dots \supset M_r = G(\mathfrak{N}\mathfrak{F}) \cdot D$, where each M_i is $\mathfrak{F}$ -abnormal in M_{i-1} and reduces Σ , $1 \leq i \leq r$. Apply Theorem 1.2 to this chain and get $D(\Sigma \cap G(\mathfrak{N}\mathfrak{F}) \cdot D) \supseteq D$.

Given a group G and a Sylow system Σ of G , we construct two sequences $\{H_i\}$ and $\{D_i\}$ as follows:

$$H_0 = G, \quad D_0 = D(\Sigma \cap H_0), \quad \text{and} \quad H_i = H_{i-1}(\mathfrak{N}\mathfrak{F}) \cdot D_{i-1}, \quad D_i = D(\Sigma \cap H_i),$$

for $i \geq 1$. We have $H_0 \supseteq H_1 \supseteq H_2 \supseteq \dots$, and $D_0 \subseteq D_1 \subseteq D_2 \subseteq \dots$ by Lemma 2.1 (b). Since $H_i(\mathfrak{N}\mathfrak{F})$ is the smallest normal subgroup of $H_i(\mathfrak{F})$ with a nilpotent factor $H_i(\mathfrak{F})/H_i(\mathfrak{N}\mathfrak{F})$, the H -sequence terminates at $H_r \in \mathfrak{F}$, if $H_{r-1} \in \mathfrak{N}\mathfrak{F}$. Then the D -sequence terminates at $D_{r-1} = H_r$.

THEOREM 2.1. $D_{r-1} = H_r$ is an $\mathfrak{F}$ -projector of G which reduces Σ .

PROOF. Since $H_i/H_i(\mathfrak{N}\mathfrak{F}) \in \mathfrak{N}\mathfrak{F}$, H_{i+1} contains an $\mathfrak{F}$ -projector of H_i by Lemma 1.3 (a). Therefore the $\mathfrak{F}$ -projector of H_{i+1} are the $\mathfrak{F}$ -projectors of H_i . Thus H_r is an $\mathfrak{F}$ -projector of G . H_r reduces Σ by the construction of the sequence.

COROLLARY 2.1. Any Sylow system Σ reduces into exactly one $\mathfrak{F}$ -projector [1, Theorem 6].

PROOF. Suppose that Σ reduces into an $\mathfrak{F}$ -projector V . Then $V \supseteq D_0$ by Lemma 2.1 (a). Since $H_1/H_0(\mathfrak{N}\mathfrak{F})$ is an $\mathfrak{F}$ -projector of $H_0/H_0(\mathfrak{N}\mathfrak{F})$, we have $D_0 \subseteq V \subseteq H_1$. Suppose that $D_{i-1} \subseteq V \subseteq H_i$. As $V \subseteq H_i$ and reduces Σ , we get $V \supseteq D_i$. Similarly, since $H_{i+1}/H_i(\mathfrak{N}\mathfrak{F})$ is an $\mathfrak{F}$ -projector of $H_i/H_i(\mathfrak{N}\mathfrak{F})$, we have $V \subseteq H_{i+1}$. Therefore $D_{r-1} = V = H_r$ is the unique $\mathfrak{F}$ -projector reducing Σ .

COROLLARY 2.2. If $G \in \mathfrak{NN}\mathfrak{F}$ then each $\mathfrak{F}$ -normalizer is contained in only one $\mathfrak{F}$ -projector of G [4, Theorem 5.9].

PROOF. Let $G \in \mathfrak{NN}\mathfrak{F}$. Then $G(\mathfrak{F}) \in \mathfrak{NN}$, $H_1 = G(\mathfrak{N}\mathfrak{F}) \cdot D_0 \in \mathfrak{N}\mathfrak{F}$, and $H_2 = D_1 \in \mathfrak{F}$. Suppose that D_0 is contained in an $\mathfrak{F}$ -projector V . Then V is contained in $G(\mathfrak{N}\mathfrak{F}) \cdot D_0$. Since $G(\mathfrak{N}\mathfrak{F})$ is a nilpotent normal subgroup of H_1 , $\Sigma \cap H_1$ reduces into $V \cap G(\mathfrak{N}\mathfrak{F})$. Hence Σ reduces into $[V \cap G(\mathfrak{N}\mathfrak{F})] \cdot D_0 = V$, by [2, Lemma 2.7]. Then $V = D_1$, by Corollary 2.1.

COROLLARY 2.3. Suppose $G \in \mathfrak{NN}\mathfrak{F}$. Let V be an $\mathfrak{F}$ -projector containing the $\mathfrak{F}$ -normalizer D of a Sylow system Σ of G . If V is self-centralizing, then $V \supseteq K$, for every subgroup K such that D is subnormal in K [4, Theorem 5.10].

PROOF. Let $D = K_0 \subset K_1 \subset \cdots \subset K_s = K$ be a subnormal series joining D and K . Suppose $K_i \subseteq V$. Since $D \subseteq K_i = K_i^x$, for any $x \in K_{i+1}$, $D \subseteq V^x$. By Corollary 2.2, $V = V^x$. If V is self-normalizing then $x \in V$. Therefore $K_{i+1} \subset V$. Hence $K \subseteq V$.

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