SUBSEMIGROUPS OF AN AMENABLE GROUP

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ABSTRACT. The following example is presented. G is a nondiscrete locally compact amenable group. H is a subgroup of G with zero Haar measure such that if $g \in G \sim H$ then (i) there is a subsemigroup $S = \bigcup_{n=0}^{\infty} I_n$ of G where the I_n 's are open pairwise disjoint ideals of S (all right- or all left-ideals) and (ii) $g \in I_0$.

The existence of an amenable discrete group containing a non-amenable subsemigroup was first considered by Frey [2]. In [1], Appel and Djorup demonstrated that a group generated by a free, and hence nonamenable subsemigroup on two generators need not be free. More recently, Hochster [5] constructed an amenable group having a nonamenable subsemigroup. In this note we present a nondiscrete locally compact amenable group G containing many open subsemigroups that are nonamenable. Specifically, G contains a subgroup H of zero Haar measure such that if $g \in G \sim H$ then (i) there is a subsemigroup $S = \bigcup_{n=0}^{\infty} I_n$ of G such that the I_n 's are open pairwise disjoint ideals of S (all right- or all left-ideals) and (ii) $g \in I_0$. [For a survey of amenability see Greenleaf [3].]

Let G be the affine group of the line, i.e.

$$G = \{g: R \rightarrow R \mid g(x) = ax + b, a \neq 0\}$$

with composition as multiplication. [The reader is referred to Hewitt and Ross [4] for the standard topology and Haar measure of G .] G , being the semidirect product of $R \sim \{0\}$ with R , is amenable.

Let $H = \{g \in G \mid g(x) = ax + b, |a| = 1\}$. H is certainly a subgroup of G , and the measure of H is zero.

Let $g_0 \in G \sim H$. Then $g_0(x) = ax + b$ where $|a| \neq 1$. Assume $|a| < 1$. Set $c = b/(1 - a)$, choose $\delta > 0$, and let $A = [c - \delta, c + \delta]$. We have $g_0(c \pm \delta) = c \pm a\delta \subset \text{int}(A)$, hence $g_0(A) \subset \text{int}(A)$. Thus there exist $U_0, V_0 \in \text{int}(A)$ such that $g_0(A) \subset (U_0, V_0)$. Choose the sequences $\langle U_i \rangle, \langle V_i \rangle$ in $\text{int}(A) \sim [U_0, V_0]$ so that $(U_i, V_i) \cap (U_j, V_j) = \emptyset$ if $i \neq j$.

Let $I_n = \{g \in G \mid g(A) \subset (U_n, V_n)\}$ for $n = 0, 1, 2, \dots$. Certainly I_n is open in G for each n , and since $(U_n, V_n) \subset A$, each I_n is a subsemi-

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group of G . Let $S = \bigcup_{n=0}^{\infty} I_n$. If $g \in I_n$ and $g' \in S$ then $gg'(A) \subset g(A) \subset (U_n, V_n)$. Hence $gg' \in I_n$. Therefore S is a semigroup and each I_n is a right-ideal of S . It is obvious that $I_n \cap I_m = \emptyset$ if $n \neq m$ and that $g_0 \in I_0$.

Let $g_0(x) = ax + b$ where $|a| > 1$. Since $g_0^{-1}(x) = a^{-1}(x - b)$, and $|a|^{-1} < 1$, the desired semigroup $S = \bigcup_{n=0}^{\infty} I_n$, containing g_0^{-1} , exist. But then $g_0 \in S^{-1} = \bigcup_{n=1}^{\infty} I_n^{-1}$ and the I_n^{-1} 's are open disjoint left-ideals of S^{-1} .

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