ON ABSTRACT PRUEFER TRANSFORMATIONS

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ABSTRACT. The Pruefer transformation has been generalized to matrix differential equations by John Barrett, and Barrett's results have been partly extended to the case of functions which take values in a B^*-algebra by Einar Hille. By modifying Barrett's proof, generalizations in the B^*-algebra case become possible.

The Pruefer transformation for second order Sturm-Liouville equations was generalized to second order matrix differential equations by Barrett [1] and subsequently studied by Reid [2] and Etgen [3]. Hille [4] considered the case of differential equations involving functions which take their values in a B^*-algebra $\mathfrak{B}$ and obtained a partial extension of Barrett's results in this more general setting. The purpose of this note is to modify Barrett's original proof in such a way that it leads to generalizations of Hille's results.

As in [2], instead of considering a differential equation of Sturm-Liouville type, we shall consider the more general first order system

$$Y' = G(x)Z, \qquad Z' = -F(x)Y, \tag{1}$$

where $G(x)$ and $F(x)$ are selfadjoint strongly continuous $\mathfrak{B}$-valued functions on some interval $[a, \infty)$ and derivatives are taken in the strong topology. We shall consider a solution Y, Z of (1) satisfying

$$Y(a) = 0, \qquad Z(a) \text{ nonsingular.} \tag{2}$$

Since (1) implies that $(Y^*Z - Z^*Y)' = 0$, we have $Y^*Z - Z^*Y = \text{constant}$ and, in view of $Y(a) = 0$,

$$Y^*Z - Z^*Y = 0. \tag{3}$$

Thus our solution pair is conjoined in the terminology of Reid [2].

The Pruefer transformation for (1) consists of determining generalized sines and cosines $S(x)$ and $C(x)$ and a nonsingular $\mathfrak{B}$-valued function $R(x)$ such that the above solution Y, Z of (1) admits the representation

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$$Y(x) = S^*(x)R(x), \qquad Z(x) = C^*(x)R(x). \tag{4}$$

The generalized sines and cosines are obtained as solutions $S(x; a, Q)$ and $C(x; a, Q)$ of

$$S' = QC, \qquad C' = -QS, \tag{5}$$

satisfying $S(a) = 0$, $C(a) = E$, where E is the identity element of $\mathfrak{B}$ and $Q(x)$ is a selfadjoint strongly continuous function of x. It can be shown [4, Chapter 9.6] that for such solutions

$$SS^* + CC^* = S^*S + C^*C = E, \tag{6}$$

$$SC^* - CS^* = S^*C - C^*S = 0. \tag{7}$$

The equations (1) and (4) yield

$$(S^*R)' = GC^*R, \qquad (C^*R)' = -FS^*R.$$

Using (5), (6), and (7), these can be solved to yield

$$R' = (SGC^* - CFS^*)R, \tag{8}$$

$$QR = (CGC^* + SFS^*)R. \tag{9}$$

Since $Z(a) = C^*(a)R(a) = R(a)$ and $Z(a)$ is nonsingular, (8) implies (see [4, p. 485]) that $R(x)$ is nonsingular for $a \leqq x < \infty$, and (9) becomes

$$Q = CGC^* + SFS^*. \tag{9'}$$

This calculation shows that (8) and (9′) are necessary conditions for the representation (4) to be valid, and a direct calculation [1] also shows that (8) and (9′) are sufficient. However, the difficulty is that since S and C both depend on Q, it is not obvious that (9′) has a solution. Indeed a major portion of Barrett's original proof consists of an existence theorem for (9′) which is established by iteration. This proof depends heavily on the fact that the $n \times n$ matrices $S(x)$ and $C(x)$ are bounded in the Frobenius-Wedderburn norm whiich assigns $\sqrt{n}$ to the unit matrix. It is in this connection that a direct generalization to the B^*-algebra case fails (see [4, p. 483]).

The question of the solvability of (9) can be circumvented, however, by defining $S(x)$ and $C(x)$ to be solutions of

$$S' = (CGC^* + SFS^*)C, \qquad C' = -(CGC^* + SFS^*)S \tag{5'}$$

satisfying $S(a) = 0$, $C(a) = E$. Our modification of Barrett's technique consists of the following observation.

THEOREM 1. *If* $R(x)$, $S(x)$ *and* $C(x)$ *satisfy* (8) *and* (5′), *respectively, then the Pruefer transformation* (4) *is valid.*

PROOF. We must verify that if (8) and (5′) are satisfied, then

$$(S^*R)' = GC^*R \quad \text{and} \quad (C^*R)' = -FS^*R.$$

The first of these follows from the direct computation

$$\begin{aligned} S^{*\prime}R + S^*R' &= C^*(CGC^* + SFS^*)R + S^*(SGS^* - CFC^*)R \\ &= (C^*C + S^*S)GC^*R = GC^*R, \end{aligned}$$

and the second follows from a similar computation.

The question remains as to whether (8) and (5′) have solutions. Since (5′) is just a first order differential equation in the Banach space $\mathfrak{B} \oplus \mathfrak{B}$ with norm

$$\left\| \begin{pmatrix} S \\ C \end{pmatrix} \right\| = |S| + |C|,$$

it follows from the generalized Picard existence theorem [**5**, Theorem 3.4.1] that (5′) has a *local* solution at $x = a$. Given $S(x)$ and $C(x)$ defined by (5′), (8) can clearly be solved for $R(x)$. This shows the existence of a local Pruefer transformation regardless of whether $S(x)$ and $C(x)$ are bounded in norm.

In order to obtain a Pruefer transformation valid on $[a, \infty)$, Hille makes the additional assumption that $Q(x)$ commutes with $\int_a^x Q(t)dt$ for all a and x. With this hypothesis he shows that $|C| \leqq 1$ and $|S| \leqq 1$, and this fact leads to an existence proof for (5). The following theorem establishes other criteria for the boundedness of $|C|$ and $|S|$.

THEOREM 2. *If $\mathfrak{B}$ is a C^*-algebra (that is an algebra of operators on a given Hilbert space $\mathfrak{H}$), then $|C| \leqq 1$ and $|S| \leqq 1$.*

PROOF. In the Hilbert space $\mathfrak{H}$, consider any solution y, z of the system

$$y' = Qz, \qquad z' = -Qy \tag{10}$$

satisfying $y(a) = 0$ and $\|z(a)\| = 1$. A direct calculation shows that

$$[(y, y) + (z, z)]' = 0,$$

so that by the initial conditions

$$(y, y) + (z, z) \equiv 1. \tag{11}$$

Now suppose that Y, Z in $\mathfrak{B}(\mathfrak{H})$ satisfies (5) and $Y(a) = 0$, $Z(a) = E$. To show that $|Y| \leqq 1$ and $|Z| \leqq 1$ for all x, we note that for any constant vector e of unit norm the vectors $y(x) = Y(x)e$, $z(x) = Z(x)e$

satisfy (10), so that by (11)

$$\|Y(x)e\| \leqq 1, \qquad \|Z(x)e\| \leqq 1$$

for any $e \in \mathfrak{H}$ satisfying $\|e\| = 1$. This completes the proof.

We remark that the above theorem applies to all complex B^*-algebras since they are known to be representable as C^*-algebras. It also applies to the real matrix case considered by Barrett [1].

Our final observation is that the boundedness of $|S|$ and $|C|$ leads to global solutions directly from (8) and (5′) and that it is possible to use classical existence theory to circumvent the existence theorems devised by Barrett and Hille for the system consisting of (5) and (9′).

From the generalized Picard existence theorem (see [5, p. 67]) applied to the space $\mathfrak{B} \oplus \mathfrak{B}$ it follows that (5′) has a right maximal interval of existence $[a, b)$ where $a < b \leqq \infty$. On this interval (5′) has a solution S, C but no solution exists on any interval $[a, b_1)$ with $b_1 > b$.

If (5′) has the property that all solutions S, C satisfy $|C| \leqq 1$, $|S| \leqq 1$, as is the case if $\mathfrak{B}$ is a C^*-algebra (Theorem 2), then one *necessarily* has $b = \infty$. This follows from the completeness of $\mathfrak{B} \oplus \mathfrak{B}$ which makes it possible to give a direct generalization of the standard extension theorem for finite systems. (See [6, p. 15, Theorem 4.1].)

The above considerations are summarized in the following theorem and corollary.

THEOREM 3. *If for any $c > a$ every solution of* (5′) $(S(a) = 0,\ C(a) = E)$ *on* $[a, c)$ *satisfies* $|S(x)| \leqq 1$ *and* $|C(x)| \leqq 1$, *then there exists a solution of* (5′) $(S(a) = 0,\ C(a) = E)$ *on the interval* $[a, \infty)$.

COROLLARY. *If* $\mathfrak{B}$ *is a C^*-algebra then there exists a solution of* (5′) $(S(a) = 0,\ C(a) = E)$ *on* $[a, \infty)$.

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