$\mathbb{R}^3$ ADMITTING A CERTAIN EMBEDDING OF P^2 IS A PSEUDO P^3

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ABSTRACT. Let M be a 3-manifold and P^2 projective 2-space. In this paper it is shown that if there exists an embedding $f: P^2 \rightarrow M$ such that $f_*: \pi_2(P^2) \rightarrow \pi_2(M)$ is trivial, then M is, except for a fake cell, projective 3-space.

In this note all spaces will be simplicial complexes and all maps will be piecewise linear. Let P^n be the n -dimensional projective space and M a 3-manifold which is connected but not necessarily compact or closed. Let $\bar{P}^3$ be the 3-manifold obtained by identifying antipodal points on the boundary of a homotopy 3-cell.

It is the purpose of this note to prove the following theorem.

THEOREM. *Let M, P^2 be as above. Let $f: P^2 \rightarrow M$ be an embedding such that $f_*: \pi_2(P^2) \rightarrow \pi_2(M)$ is trivial. Then M is a $\bar{P}^3$.*

PROOF. We claim that $f_*: \pi_1(P^2) \rightarrow \pi_1(M)$ is an injection. This may be seen as follows: Let R be a regular neighborhood of $f(P^2)$ in M . Then R is homeomorphic either to $P^2 \times I$ where I is the unit interval or to the closure of the complement of a 3-ball in P^3 .

If R is homeomorphic to $P^2 \times I$ and f_* is not 1-1, it follows from the loop theorem [1] that there is a disk $\mathcal{D}$ embedded in M which meets R only in its boundary such that the boundary of $\mathcal{D}$ is not nullhomotopic in R . Now a regular neighborhood of $\mathcal{D}$ in the closure of $M - R$ is a ball having a mobius band on its boundary! This is impossible. In the second case it follows from Van Kampen's theorem that $\pi_1(R) \rightarrow \pi_1(M)$ is an injection as is $f_*\pi_1(P^2) \rightarrow \pi_1(R)$. The claim follows.

Let (M^*, q) be the universal cover of M and construct the following commutative diagram where q_1 is a covering map and S^2 is the 2-sphere

$$\begin{array}{ccc} S^2 & \xrightarrow{g} & M^* \\ \downarrow q_1 & & \downarrow q \\ P^2 & \xrightarrow{f} & M \end{array}$$

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Since $f_{*1}:\pi_1(P^2)\rightarrow\pi_1(M)$ is an injection, $g:S^2\rightarrow M^*$ is an embedding. Since $g(S^2)$ is homotopic to a point in M^* , $g(S^2)$ bounds a finite chain in $\mathcal{C}_3(M^*; \mathbb{Z}_2)$ and thus $f(S^2)$ bounds a compact 3-submanifold B of M^* . It is a consequence of Van Kampen's theorem that B is simply connected. It is well known that this implies that B is a homotopy 3-cell.

Now $q^{-1}f(P^2)\cap B$ is a finite collection of 2-spheres each bounding a homotopy 3-cell contained in B . Thus we may choose $g(S^2)$ to bound a homotopy 3-cell B in M^* such that $q^{-1}f(P^2)\cap B = g(S^2)$.

Therefore the restriction of q to the interior of B is 1-1. But now the desired result is clear when we recognize that M is the space formed by identifying antipodal points on the boundary of B .

REFERENCE

1. J. Stallings, *On the loop theorem*, Ann. of Math. (2) **72** (1960), 12-19. MR 22 #12526.

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