AN UPPER ASYMPTOTIC ESTIMATE OF BROWNIAN PATH VARIATION

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ABSTRACT. Let $X(t, \omega)$ be standard Brownian motion, and denote by $\mathfrak{P}_n$ the family of all partitions of $[0, 1]$ with minimum distance between successive partition points $\geq 1/n$. Then

$$P \left[\limsup_{n \rightarrow \infty} \sup_{P \in \mathfrak{P}_{2n}} \frac{\log \log n}{\log^2 n} \sum_{t_i \in P} (X(t_i) - X(t_{i-1}))^2 \leq K \right] = 1$$

where K is an appropriate constant.

Let $\{X(t, \omega); 0 \leq t \leq 1\}$ be standard Brownian motion. As is customary, we will suppress the ω and write $X(t)$ for $X(t, \omega)$ and $[\dots]$ for $\{\omega | \dots\}$. Paul Lévy has shown [3] that for a sequence of refining partitions $\{P_n\}$ of $[0, 1]$ which becomes dense in $[0, 1]$

$$P \left[\lim_{n \rightarrow \infty} \text{var}(X, P_n) = 1 \right] = 1,$$

where for $P = (t_0, t_1, \dots, t_m); 0 \leq t_0 < t_1 < \dots < t_m \leq 1; \text{var}(X, P) = \sum_{i=1}^m (X(t_i) - X(t_{i-1}))^2$. For a martingale proof of this fact, see J. L. Doob [1, pp. 395ff.].

On the other hand, if the refining and density conditions on the partitions are dropped, then [3]

$$P \left[\sup_{P \in \mathfrak{P}} \text{var}(X, P) = \infty \right] = 1,$$

where $\mathfrak{P}$ is the family of all finite partitions of $[0, 1]$.

Let $\mathfrak{P}_n$ be the family of all partitions of $[0, 1]$ with minimum distance between successive partition points $\geq 1/n$. Recently, P. E. Greenwood has shown [2] that

$$P \left[\limsup_{n \rightarrow \infty} \sup_{P \in \mathfrak{P}_{2n}} \frac{1}{n^2} \text{var}(X, P) \leq K \right] = 1$$

for an appropriate constant K . It is the purpose of this note to sharpen this result by proving the following

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THEOREM.

$$P \left[\limsup_{n \rightarrow \infty} \sup_{P \in \mathfrak{P}_{2n}} \frac{\log \log n}{\log^2 n} \text{var}(X, P) \leq K \right] = 1$$

for an appropriate constant K .

REMARKS. The method of proof of our theorem parallels that of [2], with the improvement following from the sorting of the intervals Δt_i according to their size. An immediate improvement of the result in [2] can be realized by using the Schwarz inequality to replace the n^2 by n in Lemma 4, allowing $1/n$ to replace $1/n^2$ in the theorem of [2]. It would be interesting to learn how good our estimate is, perhaps by developing a good lower asymptotic estimate. This seems to be a difficult problem.

PROOF. Suppose $P \in \mathfrak{P}_{2n}$ with partition points $0 = t_0 < t_1 < \dots < t_r = 1$, and set $\Delta t_i = t_i - t_{i-1}$. For $n > e^e$, define classes $C_1, \dots, C_{\psi(n)}$ as follows:

$$\begin{aligned} C_1 &= \left\{ \Delta t_i : \frac{1}{2[\log n]} < \Delta t_i \leq 1 \right\}, \\ C_2 &= \left\{ \Delta t_i : \frac{1}{2[\log^2 n]} < \Delta t_i \leq \frac{1}{2[\log n]} \right\}, \\ &\vdots \\ C_k &= \left\{ \Delta t_i : \frac{1}{2[\log^k n]} < \Delta t_i \leq \frac{1}{2[\log^{k-1} n]} \right\}, \\ &\vdots \\ C_{\psi(n)} &= \left\{ \Delta t_i : \frac{1}{2n} \leq \Delta t_i \leq \frac{1}{2[\log^{\psi(n)-1} n]} \right\}, \end{aligned}$$

where $\psi(n) = [\log/(\log \log n)] + 1$. Then

$$\frac{1}{[\log^{\psi(n)} n]} < \frac{1}{n} \leq \frac{1}{[\log^{\psi(n)-1} n]}.$$

The 2 in the denominators above are for convenience of estimations later on.

Now, setting

$$\varphi(n) = \log^2 n / (\log \log n),$$

$$\begin{aligned}
 & P[\text{var}(P, X) > K\varphi(n)] \\
 (1) \quad & = P\left[\sum_{k=1}^{\psi(n)} \sum_{C_k} (X(t_i) - X(t_{i-1}))^2 > K\varphi(n)\right] \\
 & \leq \sum_{k=1}^{\psi(n)} P\left[\sum_{C_k} (X(t_i) - X(t_{i-1}))^2 > K\varphi(n)\psi^{-1}(n)\right].
 \end{aligned}$$

The symbol $\sum_{C_k}$ means the sum over all those indices i with $\Delta t_i \in C_k$.

Consider the inside sum $\sum_{C_k}$ of (1). Break $[0, 1]$ into equal subintervals of the form

$$\left[\frac{(l-1)}{2[\log^k n]}, \frac{l}{2[\log^k n]}\right] = [p_{l-1}^{(k)}, p_l^{(k)}] \quad \text{for } l = 1, 2, \dots, 2[\log^k n].$$

No more than $\log n + 1$ points of the form $p_l^{(k)}$ lie in any Δt_i of C_k for $k = 2, 3, \dots, \psi(n)$, and less than $2 \log n$ points of the form $p_l^{(1)}$ lie in any Δt_i of C_1 . Fix k and suppose for some Δt_i in C_k that $t_{i-1} \leq p_i^{(k)} < p_{i+1}^{(k)} < \dots < p_{i+m_i}^{(k)} \leq t_i$. Then, with $n > e^e$, so that $2 \log n > \log n + 1$,

$$\begin{aligned}
 & (X(t_i) - X(t_{i-1}))^2 \\
 (2) \quad & \leq 2 \log n [(X(t_i) - X(p_{i+m_i}^{(k)}))^2 + (X(p_{i+m_i}^{(k)}) - X(p_{i+m_i-1}^{(k)}))^2 \\
 & \quad + \dots + (X(p_{i+1}^{(k)}) - X(t_{i-1}))^2],
 \end{aligned}$$

since by the Schwarz inequality, $(a_r - a_1)^2 \leq r \sum_2^r (a_i - a_{i-1})^2$ for any r real numbers $a_1, a_2, \dots, a_r$. We sum the left-hand side of (2) over all Δt_i in C_k . On the right-hand side we replace the existing t_i 's by a point t in the same interval $[p_{i-1}^{(k)}, p_i^{(k)}]$ so as to maximize $(X(p_i^{(k)}) - X(t))^2 + (X(t) - X(p_{i-1}^{(k)}))^2$; and we insert such a point t into all those intervals with no t_i in them. Then we sum the right-hand side over all the points $p_i^{(k)}$ and t , and we get

$$\begin{aligned}
 & \sum_{C_k} (X(t_i) - X(t_{i-1}))^2 \\
 (3) \quad & \leq 2 \log n \sum_{l=1}^{2[\log^k n]} \sup_{t \in [p_{l-1}^{(k)}, p_l^{(k)}]} [(X(p_l^{(k)}) - X(t))^2 + (X(t) - X(p_{l-1}^{(k)}))^2].
 \end{aligned}$$

The last expression is independent of the partition P we started with. The class C_k could have been empty to start with, but the inequality holds nevertheless.

Now we have, from (1) and (3),

$$\begin{aligned}
 & P \left[\sup_{P \in \mathfrak{P}_{2n}} \text{var}(X, P) > K\varphi(n) \right] \\
 &= P \left[\sup_{P \in \mathfrak{P}_{2n}} \sum_{k=1}^{\psi(n)} \sum_{C_k} (X(t_i) - X(t_{i-1}))^2 > K\varphi(n) \right] \\
 &\leq \sum_{k=1}^{\psi(n)} P \left[\sup_{P \in \mathfrak{P}_{2n}} \sum_{C_k} (X(t_i) - X(t_{i-1}))^2 > K\varphi(n)\psi^{-1}(n) \right] \\
 (4) \quad &\leq \sum_{k=1}^{\psi(n)} P \left[2 \log n \sum_{l=1}^{2[\log^k n]} \sup_{t \in [p_{l-1}^{(k)}, p_l^{(k)}]} [(X(p_l^{(k)}) - X(t))^2 \right. \\
 &\quad \left. + (X(t) - X(p_{l-1}^{(k)}))^2] > K_1 \log n \right]
 \end{aligned}$$

where K_1 is a constant with $K\varphi(n)\psi^{-1}(n) > K_1 \log n$. Since $n > e^e$ we can take $K_1 = K/2$.

For integers k , $1 \leq k \leq \psi(n)$, set

$$(5) \quad Y_{k,l} = \sup_{t \in [p_{l-1}^{(k)}, p_l^{(k)}]} [(X(p_l^{(k)}) - X(t))^2 + (X(t) - X(p_{l-1}^{(k)}))^2].$$

For each k , the $Y_{k,l}$, $l=1, 2, \dots, 2[\log^k n]$, are independent, identically distributed random variables. By standard Brownian motion arguments (see Lemmas 1 and 2 of [2]) one can show that

$$(6) \quad P[Y_{k,l} > x] \leq 2P[8(X(p_l^{(k)}) - X(p_{l-1}^{(k)}))^2 > x]$$

and $(X(p_l^{(k)}) - X(p_{l-1}^{(k)}))^2$, $l=1, 2, \dots, 2[\log^k n]$, are independent, identically distributed variables. Putting (6) another way, if the $Y_{k,l}$ have F_k as distribution function, and the common distribution of the variables $8(X(p_l^{(k)}) - X(p_{l-1}^{(k)}))^2$ is G_k , then $1 - F_k(x) \leq 2(1 - G_k(x))$ for each k and all x . By induction (see Lemma 3 in [2]), for a positive integer m ,

$$(7) \quad 1 - F_k^{m*}(x) \leq 2^m(1 - G_k^{m*}(x)).$$

The variables $[\log^k n]^{1/2}(X(p_l^{(k)}) - X(p_{l-1}^{(k)}))$, $l=1, 2, \dots, 2[\log^k n]$, are independent, normally distributed each with mean 0 and variance $\frac{1}{2}$. Hence the sum of their squares has a χ^2 -type distribution and we get from (6) and (7) that

$$\begin{aligned}
 (8) \quad & P \left[\sum_{l=1}^{2[\log^k n]} Y_{k,l} > x \right] \\
 & \leq 2^{2[\log^k n]} P \left[8 \sum_{l=1}^{2[\log^k n]} (X(p_l^{(k)}) - X(p_{l-1}^{(k)}))^2 > x \right] \\
 & = \frac{2^{2[\log^k n]}}{\Gamma([\log^k n])} \int_{(x/8)[\log^k n]}^{\infty} t^{[\log^k n]-1} e^{-t} dt.
 \end{aligned}$$

Then, by (4), (5) and (8),

$$\begin{aligned}
 (9) \quad & P \left[\sup_{P \in \mathfrak{P}_{2n}} \text{var}(X, P) > K\varphi(n) \right] \\
 & \leq \sum_{k=1}^{\psi(n)} P \left[\sum_{l=1}^{2[\log^k n]} Y_{k,l} > \frac{K_1}{2} \right] \\
 & \leq \sum_{k=1}^{\psi(n)} \frac{2^{2[\log^k n]}}{\Gamma([\log^k n])} \int_{K_2[\log^k n]}^{\infty} t^{[\log^k n]-1} e^{-t} dt
 \end{aligned}$$

where $32K_2 = K$. If we could show that the last expression in (9) is the n th term of a convergent series, then by the convergence part of the Borel-Cantelli Lemma, the event $[\sup_{P \in \mathfrak{P}_{2n}} \text{var}(X, P) > K\varphi(n)]$ would occur only finitely often with probability one, and our theorem would be proven.

We estimate the integral in (9) by repeatedly integrating by parts. For fixed k , and setting $m = [\log^k n]$ for notational convenience, we get

$$\begin{aligned}
 \int_{K_2 m}^{\infty} t^{m-1} e^{-t} dt &= e^{-K_2 m} (K_2 m)^{m-1} \left(1 + \frac{m-1}{K_2 m} + \dots + \frac{(m-1)!}{(K_2 m)^{m-1}} \right) \\
 &\leq e^{-K_2 m} (K_2 m)^{m-1} \frac{m!}{m^m} \sum_{l=0}^m \frac{m^l}{l!} \quad \text{for } K_2 \geq 1.
 \end{aligned}$$

Now² $\sum_{l=0}^m (m^l/l!) \sim \frac{1}{2} e^m$, and it follows that, for $K_2 \geq 1$,

$$\begin{aligned}
 (10) \quad & \frac{2^{2m}}{\Gamma(m)} \int_{K_2 m}^{\infty} t^{m-1} e^{-t} dt \leq \frac{2^{2m}}{m!} e^{-K_2 m} (K_2 m)^m \frac{m!}{m^m} \sum_{l=0}^m \frac{m^l}{l!} \\
 & \leq \exp\{(2 \log 2 - K_2 + 1 + \log K_2)m\}
 \end{aligned}$$

² This relation can be seen by using the Central Limit Theorem for a sequence of independent, identically distributed Poisson random variables with parameter 1. Also, the relation follows from the theory of Szász operators [4].

for large m , i.e. for large n . We take K_2 large enough so that $2 \log 2 - K_2 + 1 + \log K_2 < -1$, and $n_0 > e^e$ and large enough so that the estimates (10) hold for $n > n_0$. Then

$$\begin{aligned} \sum_{n=n_0}^{\infty} \sum_{k=1}^{\psi(n)} \frac{2^{2[\log^k n]}}{\Gamma([\log^k n])} \int_{K_2[\log^k n]}^{\infty} t^{[\log^k n]-1} e^{-t} dt \\ \leq \sum_{n=n_0}^{\infty} \psi(n) \exp\{(2 \log 2 - K_2 + 1 + \log K_2)[\log n]\} < \infty, \end{aligned}$$

and this proves our theorem.

REFERENCES

1. J. L. Doob, *Stochastic processes*, Wiley, New York and Chapman & Hall, London, 1953. MR 15, 445.
2. P. E. Greenwood, *An asymptotic estimate of Brownian path variation*, Proc. Amer. Math. Soc. 21 (1969), 134-138. MR 38 #3920.
3. P. Lévy, *Le mouvement brownien plan*, Amer. J. Math. 62 (1940), 487-550. MR 2, 107.
4. O. Szász, *Generalization of S. Bernstein's polynomials to the infinite interval*, J. Res. Nat. Bur. Standards 45 (1950), 239-245; Collected Math. Works (Cincinnati, 1955), 1401-1407. MR 13, 648.

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