ON $\mathfrak{F}$ -NORMALIZERS AND $\mathfrak{F}$ -HYPERCENTER

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ABSTRACT. In this note, we shall prove a theorem which is a generalization of the following theorem: Let G be a soluble group, then the intersection of all system normalizers of G is the hypercenter of G .

Recently, Carter and Hawkes [1] have generalized the construction of system normalizers of finite soluble groups introducing the concept of the $\mathfrak{F}$ -normalizer, and B. Huppert [2] has generalized the hypercenter introducing the concept of the $\mathfrak{F}$ -hypercenter. In this note, we shall show the relations between $\mathfrak{F}$ -normalizers and the $\mathfrak{F}$ -hypercenter, which generalize the theorem [3, VI. 11.11] on the relations between the system normalizers and the hypercenter. The methods of proof of our theorem are similar to those in [3].

1. DEFINITIONS. All groups in this note are soluble and finite.

If nonempty formations $\mathfrak{F}(p)$, one for each p , are given, the local formation $\mathfrak{F}$ locally defined by $\{\mathfrak{F}(p)\}$ is the class of all groups G such that, whenever M is a chief factor of G , of order p^n , say, then the automorphisms group induced on M by G belongs to $\mathfrak{F}(p)$. And M is called an $\mathfrak{F}$ -central- p -chief factor. By Carter and Hawkes [1, p. 177] we can choose $\mathfrak{F}(p)$ such that $\mathfrak{F}(p) \subset \mathfrak{F}$. Let $\{S^p\}$ be a set of p -complements of G , one for each prime p dividing $|G|$, and let γ be the Sylow system of G generated by $\{S^p\}$. We write $T^p = S^p \cap C_p$ for each prime p dividing $|G|$, where C_p is the intersection of the centralizers of the $\mathfrak{F}$ -central- p -chief factors of G . The set $T = \{T^p\}$ will be called an $\mathfrak{F}$ -system of G (see [1]). And then the subgroup $D = \bigcap_p N_G(T^p)$ will be called the $\mathfrak{F}$ -normalizer of G (see [1]). Since any two Sylow systems of G are conjugate in G , it follows that the $\mathfrak{F}$ -normalizers of G form a characteristic conjugacy class of subgroups of G . Let N be a normal subgroup of G with a normal chain $e = N_0 \triangleleft N_1 \triangleleft \cdots \triangleleft N_r = N$ where each N_i is normal in G . N is called an $\mathfrak{F}$ -hypercentral subgroup whenever the following conditions are satisfied

- (1) N_i/N_{i-1} is a chief factor of G .
- (2) If the order of N_i/N_{i-1} is p_i^n , then $G/C_G(N_i/N_{i-1}) \in \mathfrak{F}(p_i)$.

As a product of $\mathfrak{F}$ -hypercentral normal subgroups is $\mathfrak{F}$ -hypercentral, so the product of all $\mathfrak{F}$ -hypercentral normal subgroups of G

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is called the $\mathfrak{F}$ -hypercenter of G , say $Z_{\mathfrak{F}}$ (see [2]). If $\mathfrak{F}(p) \subset \mathfrak{F}$, then $\mathfrak{F}$ -normalizers and $\mathfrak{F}$ -hypercenter depend only on $\mathfrak{F}$ and not on $\mathfrak{F}(p)$ (see [1], [2]).

2. THEOREM. *Let G be a soluble group, then*

(a) *Each $\mathfrak{F}$ -normalizer of G is not contained in any proper normal subgroup of G .*

(b) *The subgroup being generated by all $\mathfrak{F}$ -normalizers of G is G .*

(c) *The intersection of all $\mathfrak{F}$ -normalizers of G is the $\mathfrak{F}$ -hypercenter.*

This is a generalization of [3, Theorem VI, 11.11].

PROOF. (a) If M is a maximal subgroup of G , then G/M is a central chief factor and so G/M is $\mathfrak{F}$ -central. Hence each $\mathfrak{F}$ -normalizer D of G covers on G/M by Theorem 4.1 of [1], so $D \not\subseteq M$.

(b) Since all $\mathfrak{F}$ -normalizers of G are conjugate, the subgroup being generated by all $\mathfrak{F}$ -normalizers of G is a normal subgroup containing an $\mathfrak{F}$ -normalizer of G . From (a), we see it is G .

(c) As the hypercenter $Z_{\mathfrak{F}}$ of G contains only $\mathfrak{F}$ -central chief factors of G , so $Z_{\mathfrak{F}}$ is covered by every $\mathfrak{F}$ -normalizer of G , thus $Z_{\mathfrak{F}} \subseteq \bigcap_{\theta \in G} D^{\theta}$. Set $\phi = \{T^p\}$ and $\phi^* = \{T^p Z_{\mathfrak{F}}/Z_{\mathfrak{F}}\}$. By [1, Theorem 4.1, Corollary 2], as $\mathfrak{F}$ -normalizers are homomorphic invariant,

$$N_G(N(\phi))Z_{\mathfrak{F}}/Z_{\mathfrak{F}} \subseteq N_{G/Z_{\mathfrak{F}}}(N(\phi)Z_{\mathfrak{F}}/Z_{\mathfrak{F}}) = N_{G/Z_{\mathfrak{F}}}(N(\phi^*)).$$

To prove the theorem, it is sufficient to show $\bigcap_{\theta \in G/Z_{\mathfrak{F}}} N_{G/Z_{\mathfrak{F}}}(N(\phi^*))^{\theta^*} = e$, because we see $\bigcap_{\theta \in G} N_G(N(\phi))^{\theta} Z_{\mathfrak{F}}/Z_{\mathfrak{F}} = e$ so $\bigcap_{\theta \in G} N_G(N(\phi))^{\theta} \subseteq Z_{\mathfrak{F}}$, since it is clear $\bigcap_{\theta \in G} N(\phi)^{\theta} \subseteq \bigcap_{\theta \in G} N_G(N(\phi))^{\theta}$, then $\bigcap_{\theta \in G} D^{\theta} = \bigcap_{\theta \in G} N(\phi)^{\theta} \subseteq Z_{\mathfrak{F}}$, thus $\bigcap_{\theta \in G} D^{\theta} = Z_{\mathfrak{F}}$. Hence we may assume $Z_{\mathfrak{F}} = e$. Let N be a minimal normal subgroup of G , which is contained in $N_G(D) = N_G(N(\phi))$. As $Z_{\mathfrak{F}} = e$, N is an $\mathfrak{F}$ -excentral, which is avoided by $N(\phi) = D$, by Theorem 4.1 of [1]. As $N \subseteq N_G(N(\phi))$ so that $[N, N(\phi)] \subseteq N \cap D = e$, D is centralized by N , so G generated by all D 's is centralized by N , which is a contradiction since N would be in $Z_{\mathfrak{F}}$.

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