AN ARCWISE CONNECTED DENSE HAMEL BASIS FOR HILBERT SPACE

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ABSTRACT. This paper shows if X is an infinite dimensional Banach space, X contains a linearly independent arc. Also based on the continuum hypothesis, that if X is an infinite dimensional Banach space and card $X=c$, then X contains a dense arcwise connected Hamel basis.

The main result of this paper is that any infinite dimensional Banach space with the same cardinal number as the real line contains an arcwise connected dense Hamel basis. The proof of this result uses the continuum hypothesis. Two lemmas are proved which are of independent interest. Lemma 1 shows that a large class of subspaces of normed linear spaces are of the 1st category. In this connection, Hausdorff showed that any infinite dimensional real Banach space contains a second category linear subspace which is not complete under any equivalent norm [2]. Lemma 2 deals with homeomorphisms in the space $C(I, X)$ of mappings from the unit interval I into an infinite dimensional Banach space X. It is known that this set of homeomorphisms is dense in $C(I, X)$; and G. G. Johnson [3] and [4], while working on problem 4 in Halmos' book [1] has shown that if f is a homeomorphism in $C(I, X)$ such that each two nonoverlapping chords of $f([0, 1])$ are orthogonal, then the nonzero values of f are linearly independent. It follows from Lemma 2 that the set of all homeomorphisms in $C(I, X)$ with linearly independent range is, in fact, a dense G_δ in $C(I, X)$.

LEMMA 1. *If X is an infinite dimensional normed linear space which is spanned by a subset of a σ-compact set, X is first category.*

PROOF. Let $B_1, B_2, \cdots$ be compact subsets of X so that $X=L[\bigcup_{i=1}^{\infty} B_i]$ and $B_i \subseteq B_{i+1}$. Let $M(i, j)=\{y \mid y=c_1x_1+\cdots+c_ix_i$ where $|c_k| \leqq i$ and $x_k \in B_j$ for $k \in \{1, \cdots, i\}\}$.

Because B_j is compact, $M(i, j)$ is compact. Since X is infinite dimensional, it is not locally compact. Therefore $M(i, j)$ contains no open sets. But X is the union of $M(i, j)$ over all positive i and j. Therefore X is first category.

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LEMMA 2. *If X is an infinite dimensional Banach space and X' is a linear subspace of X which has a σ-compact Hamel basis then there exists a homeomorphism $h:I\to X$ such that $L[h(I)]\cap X'=0$ and $h(I)$ is linearly independent. Furthermore, the set of continuous functions f from I into X such that $f(I)$ is linearly dependent or f is not one-to-one is first category.*

PROOF. Let $A_1, A_2, \cdots$ be a basis of connected open sets for the open sets of I. Let $B_1, B_2, \cdots$ be compact subsets of X so that $B_1\subseteq B_2\subseteq \cdots$ and the union of the B_i's contains a Hamel basis for X'. Let $M(i, j)=\{y\,|\,y=c_1x_1+\cdots+c_ix_i$ where $|c_k|\leqq i$ and $x_k\in B_j$ for $k\in\{1, \cdots, i\}\}$.

Let $M(i, j, i_1, \cdots, i_l)=\{f\in C(I, X)\,|$ there exist $y_k\in c\,|\,l\,|\,A_{i_k}$, $d_k\in[-l, l]$ such that $d_1=1$ and $d_1f(y_1)+\cdots+d_lf(y_l)$ belongs to $M(i, j)\}$ where if $c\,|\,l\,|\,A_{i_k}\cap c\,|\,l\,|\,A_{i_r}\neq\varnothing$, $r=k$.

Let $M(i, j, i_1, \cdots, i_l)=\varnothing$ if $c\,|\,l\,|\,A_{i_k}\cap c\,|\,l\,|\,A_{i_r}\neq\varnothing$ for some $r\neq k$.

It is clear $M(i, j, i_1, \cdots, i_l)$ is closed in $C(I, X)$.

Now assume $f\in M(i, j, i_1, \cdots, i_l)$.

Since $f(I)\cup B_j$ is compact, $L[f(I)\cup B_j]$ is first category. Let $\epsilon_1\in X-L[f(I)\cup B_j]$. Let D_1 be an open connected subset of R such that D_1 contains $c\,|\,l\,|\,A_{i_1}$ and $c\,|\,l\,|\,D_1\cap c\,|\,l\,|\,A_{i_j}=\varnothing$ for $j\in\{2, \cdots, l\}$.

Let d_1 be the midpoint of D_1 and r_1 the radius of D_1.

Let $f_i(x)=f(x)$ for $x\in I-D_1$.

Let $f_i(x)=f(x)+(r_1-|x-d_1|)\epsilon_1/2^i$ for $x\in I\cap D_1$.

It is clear $f_i\notin M(i, j, i_1, \cdots, i_l)$ and that $f_i\to f$.

Therefore $M(i, j, i_1, \cdots, i_l)$ is nowhere dense.

Note the union of all these sets would include all functions in $C(I, X)$ that are not one-to-one or that map I onto a linearly dependent set.

Since $C(I, X)$ is complete, there exists a point h not in that set. This function h will satisfy the conclusion.

THEOREM 3. *If X is an infinite dimensional Banach space,* card $X=c$ *and the continuum hypothesis is satisfied, then X contains a dense arcwise connected Hamel basis.*

PROOF. Let A be a Hamel basis for X. Let B be a basis for the open sets of X so that card $B=c$. By the continuum hypothesis, card B $=$card $A=c$, the smallest uncountable cardinal number. Well order A so that every element of A has at most a countable number of predecessors. Let f be a one-to-one function from A onto B. Let a_1 be the first element of A. Let $b_1\in f(a_1)-L[\{a_1\}]$. By Lemma 2, there exists a homeomorphism h'' of I into X so that $L[h''(I)]\cap L[\{a_1, b\}]$ $=0$ and $h''(I)$ is linearly independent.

Let $h'(x)=xh''(x)+(1-x)a_1$ for $x\in I$.

Let $h(x)=(1-x)h'(x)+xb_1$.

Let $A_{a_1}=h(I)$.

A_{a_1} is a linearly independent arc from a_1 to b_1.

Let $a\in A$. Assume now for all $b\in A$ such that $b<a$,

(1) A_b is a linearly independent set.

(2) A_b contains a point of $f(b)$, and $L[A_b]$ contains b.

(3) A_b is a countable union of arcs each of which contains a_1.

(4) If $c\in A$ and $c<b$, $A_c\subseteq A_b$.

Let $C=\bigcup_{b<a} A_b$. Using Lemma 1, let $f'(a)\in f(a)-L[C]$.

By Lemma 2, there exists a homeomorphism h_a'' of I into X such that $h_a''(I)$ is linearly independent and $L[h''(I)]\cap L[C\cup f'(a)]=0$.

Let $h_a'(x)=xh_a''(x)+(1-x)a_1$ for $x\in I$.

Let $h_a(x)=(1-x)h_a'(x)+xf'(a)$.

If $a\in L[C\cup h_a(I)]$, let $A_a=C\cup h_a(I)$.

If $a\notin L[C\cup h_a(I)]$, let g_a'' be a homeomorphism of I into X so that $g_a''(I)$ is linearly independent and that $L[g_a''(I)]\cap L[C\cup h_a(I)\cup a]=0$.

Let $g_a'(x)=xg_a''(x)+(1-x)a_1$ for $x\in I$.

Let $g_a(x)=(1-x)g_a'(x)+xa$ for $x\in I$.

Let $A_a=B\cup h_a(I)\cup g_a(I)$.

In either case A_a is a countable union of arcs each having a_1 as an endpoint, $a\in L[A_a]$, there is a point appearing both in $f(a)$ and A_a and A_a is linearly independent.

Therefore $H=\bigcup_{a\in A} A_a$ is a Hamel basis for X which is dense and arcwise connected.

References

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