GRAPHS WITH A LARGE CAPACITY

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ABSTRACT. A constructive method for obtaining graphs with a relatively large capacity is given. The method uses products of graphs.

Introduction. In this note we present a constructive method for obtaining graphs with a relatively large capacity and obtain an upper bound for the capacity $\theta(G)$ of a graph G. The capacity of a graph was introduced by Shannon [5], for investigations of problems concerning noisy channels in information theory. If $\mu(G)$ is the maximal number of independent vertices in the graph G, it is well known [5], that $\theta(G) \geqq \mu(G)$. Our method will yield for every $k \geqq 0$, a graph G_k such that $\theta(G_k) \geqq k \cdot \mu(G_k)$. The construction is based on the composition of graphs introduced by Harary [1].

The definitions and generally accepted notations throughout this paper will be those used in Harary [2]. $\mu(G)$ will denote the maximal number of independent vertices in G. Since the strong product and composition of graphs (for definitions see Harary [1]) are associative, powers of G with respect to each one of them is well defined. We denote those powers by G^n and $G^{[n]}$ resp. The capacity of G is defined by

$$\theta(G) = \sup_n \mu(G^n)^{1/n}.$$

Upper bound for $\theta(G)$: If A and B are independent sets in G and H resp., $A \times B$ is independent in $G \times H$, hence

$$\mu(G \times H) \geqq \mu(G) \cdot \mu(H) \Rightarrow \mu(G^n) \geqq \mu^n(G) \Rightarrow \theta(G) \geqq \mu(G).$$

To obtain an upper bound, we consider the function $\alpha(G)$ introduced in [4], and defined as follows: Let $V(G) = \{g_1, \cdots, g_n\} \cdot \{C_1, \cdots, C_s\}$ is a fixed ordering of all maximal complete subgraphs of G. $\alpha_i^j = 1$ if $g_i \in C_j$, $\alpha_i^j = 0$ otherwise

$$P(G) = \left\{(x_1, \cdots, x_n) \mid x_i \geqq 0, \sum_{i=1}^{n} \alpha_i^j x_i \leqq 1, 1 \leqq j \leqq s\right\},$$

$$\alpha(G) = \max_{x \in P(G)} \sum_{i=1}^{n} x_i, \qquad x = (x_1, \cdots, x_n).$$

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THEOREM 1. *If G is an ordinary graph then*

$$\theta(G) \leqq \alpha(G).$$

PROOF. Let H be any graph and let $A \subset V(G\times H)$ be a maximal independent set of vertices in $G\times H$ (card $A = \mu(G\times H)$). Let

$$A_i = \{h \mid (g_i, h) \in A\}, \quad A_i' = \{(g_i, h) \mid (g_i, h) \in A\}.$$

Obviously, $\{A_i'\}$ is a disjoint decomposition of A. Hence

$$\sum_{i=1}^{n} \text{card } A_i' = \mu(G \times H).$$

By the definition of A_i and the independence of A we have card A_i $=$card A_i', A_i is an independent set in H. Let

$$x_i = \text{card } A_i \cdot 1/\mu(H).$$

We will show first that $(x_1, \cdots, x_n) \in P(G)$. For simplicity of notation, we may assume without loss of generality, that $C_j = \{g_1, \cdots, g_k\}$. Since A is independent, and $(g_i, g_j) \in E(G)$, $1 \leqq i, j \leqq k$, we must have $A_i \cap A_j = \emptyset$ and $A_i \cup A_j$ is an independent set in H. By the same argument, $\bigcup_{i=1}^{k} A_i$ is an independent set in H and the union is disjoint. Hence

$$\mu(H) \sum_{i=1}^{n} \alpha_i^j x_i = \sum_{i=1}^{k} \text{card } A_i = \text{card } \bigcup_{i=1}^{k} A_i \leqq \mu(H) \Rightarrow \sum_{i=1}^{n} \alpha_i^j x_i \leqq 1.$$

Therefore $(x_1, \cdots, x_n) \in P(G)$. Hence we have

$$\alpha(G) \geqq \sum_{i=1}^{n} x_i = \frac{1}{\mu(H)} \sum_{i=1}^{n} \text{card } A_i = \frac{1}{\mu(H)} \sum_{i=1}^{n} \text{card } A_i' = \frac{\mu(G \times H)}{\mu(H)}$$

$\Rightarrow$

$$\alpha(G) \cdot \mu(H) \geqq \mu(G \times H).$$

Since $\alpha(G) \geqq \mu(G)$ [4], by induction we obtain

$$\mu(G^n) \leqq \mu(G^{n-1})\alpha(G) \leqq \alpha^n(G) \Rightarrow \theta(G) \leqq \alpha(G).$$

REMARK. In [4], it was shown that $\mu(G\times H) = \mu(G) \cdot \mu(H)$ for all graphs H iff $\alpha(G) = \mu(G)$. Hence for such graphs we have $\theta(G) = \alpha(G)$ $= \mu(G)$. Since the capacity of no other graphs is known nothing else can be said about the upper bound established above.

THEOREM 2. *For every $k > 0$, there exists a graph G_k such that $\theta(G_k) \geqq k\mu(G_k)$.*

PROOF. Let G_0 be a self-complemented graph with n vertices such that $\mu(G_0)\cdot\mu(\overline{G}_0)<n$ (e.g. a pentagon). Let p be a positive integer such that $n^p \leqq k^2\mu^{2p}(G_0)$. Obviously, such a number exists since by our assumptions $\mu^2(G_0)=\mu(G_0)\cdot\mu(\overline{G}_0)<n$. Let $G_k=G_0^{[p]}$. It is easy to see that $\mu(G[H])=\mu(G)\cdot\mu(H)$, hence $\mu(G_k)=\mu^p(G_0)$. If G and H are self-complemented, it was shown by Sabidussi that $G[H]$ is self-complemented, hence G_k is self-complemented. Consider the set $A=\{(g,g)\mid g\in V(G_k)\}$ as a subset of $V(G_k\times\overline{G}_k)$. Since $(g,g')\in E(G_k) \Rightarrow (g,g')\notin E(\overline{G}_k)$ it follows that A is an independent set of vertices in $G_k\times\overline{G}_k$. Since card $A=n^p$ we get

$$(*)\qquad \mu(G_k^2) = \mu(G_k\times\overline{G}_k) \geqq n^p.$$

Ljubič [3] has shown that $\theta(G)=\lim_n (\mu(G^n))^{1/n}$ therefore we have in general

$$\theta(G) = \lim_n(\mu(G^{2n}))^{1/2n} = \left(\lim_n \mu(G^{2n})^{1/n}\right)^{1/2} = \theta^{1/2}(G^2).$$

By using the lower bound for the capacity of a graph and (*) we obtain

$$\theta(G_k) = \theta^{1/2}(G_k^2) \geqq n^{p/2} \geqq k\mu^p(G_0) = k\mu(G^{[p]}) = k\cdot\mu(G_k).$$

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