LOWER BOUNDS TO THE ZEROS OF SOLUTIONS OF $y''+p(x)y=0$

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ABSTRACT. If $p(x)$ is nonnegative, monotonic and concave, no solution of $y''+p(x)y=0$ has more than $n+1$ zeros in the interval (a, b) defined by

$$(b-a)\int_a^b p(x)dx = n^2\pi^2.$$

This is proved by showing that, if $y'(a)=0$, the nth succeeding zero of $y'(x)$ will not precede b.

This note gives a proof of the following:

THEOREM. *Let $p(x)$ be nonnegative, monotonically nondecreasing, and concave (no point on an arc lies below the chord). Let $y(x)$ be a solution of*

$$y'' + p(x)y = 0 \tag{1}$$

whose derivative vanishes at $x=a$. Then if n is a positive integer and b is defined by

$$(b-a)\int_a^b p(x)dx = n^2\pi^2, \tag{2}$$

the derivative of any other solution of (1) *will vanish at most n times in (a, b).*

This is a companion to the theorem of [3], which has been independently generalized by Cohn [1] and by Elbert [2]. A corollary shows that no solution of (1) can have more than $n+1$ zeros in the closed interval $[a, b]$.

Throughout, $y(x)$ will be the solution of (1) satisfying $y(a)=1$, $y'(a)=0$; and in the first part of the paper n will be 1.

If $p(x)$ is constant the derivative vanishes exactly n times, so it will be assumed that $p(x)$ is increasing; if $p(x)$ decreases, the transformation $x'=-x$ reduces the problem to the case considered. Since a linear transformation on x will carry (a, b) into $(0, \pi)$, the solution $y(x)$ will be assumed to satisfy $y(0)=1$, $y'(0)=0$, $y'(\pi)=0$, with $y(x)=0$ at exactly one intermediate point $x=c$.

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It is convenient to introduce polar coordinates, as in [5], by $y=\rho\cos\theta$, $y'=-\rho\sin\theta$, so that

$$\rho' = \rho(p-1)\sin\theta\cos\theta, \tag{3}$$

$$\theta' = 1 + (p-1)\cos^2\theta. \tag{4}$$

Since $\theta(0)=0$ and $\theta(\pi)=\pi$, (4) can be integrated to give

$$\pi = \pi + \int_0^\pi (p-1)\cos^2\theta dx, \quad \text{so} \int_0^\pi (p-1)\cos^2\theta dx = 0. \tag{5}$$

It will be shown by approximating (5) that

$$\int_0^\pi (p-1)dx \geqq 0.$$

From (5), $p-1$ cannot have one sign in $(0, \pi)$. Then $p(x)=1$ at exactly one point; say $p(d)=1$, $0<d<\pi$.

Now suppose the intervals $(0, \pi/2)$ and $(\pi/2, \pi)$ of θ each divided into m equal parts $\Delta\theta$. Let the values of x corresponding to the end-points be

$$x_0=0<x_1<x_2< \cdots <x_m=c \quad \text{and} \quad X_0=\pi>X_1> \cdots >X_m=c.$$

Let $\Delta_k x = x_k - x_{k-1}$ and $\Delta_k X = X_{k-1}-X_k$. From (4),

$$\Delta\theta = \Delta_k x + \int_{x_{k-1}}^{x_k} (p-1)\cos^2\theta dx = \Delta_k x + (\bar{p}_k - 1)\cos^2\bar{\theta}_k \cdot \Delta_k x,$$

where the barred letters denote mean values.
Then

$$\Delta_k x = \Delta\theta/[1 + (\bar{p}_k - 1)\cos^2\bar{\theta}_k]. \tag{6a}$$

Similarly,

$$\Delta_k X = \Delta\theta/[1 + (\bar{P}_k - 1)\cos^2\bar{\theta}_k]. \tag{6b}$$

The capital P will mean that x is in some interval $\Delta_k X$.

THEOREM 1. $\int_0^\pi (p-1)dx \geqq 0$.

PROOF. Given any positive number ϵ, an integer m can be found such that

$$-\epsilon/2 < \int_0^c (1-p)\cos^2\theta dx - \sum_{k=1}^{m} [(1-\bar{p}_k)\cos^2\bar{\theta}_k]\Delta_k x < \epsilon/2, \tag{7a}$$

$$\text{(7b)}\quad -\epsilon/2 < \int_c^\pi (p-1)\cos^2\theta dx - \sum_{k=1}^m [(\bar{P}_k - 1)\cos^2\bar{\theta}_k]\Delta_k X < \epsilon/2,$$

where $\Delta_k x$ and $\Delta_k X$ are determined from (6a) and (6b). In fact, m can be chosen so large that, while maintaining the inequality, (7a) and (7b) can be modified by choosing $\bar{p}_k$ and $\bar{P}_k$ as the values of $p(x)$ at the left-hand ends of the intervals, and $\cos^2\bar{\theta}_k$ can be the same for $\Delta_k x$ and $\Delta_k X$. This will be assumed done, and the bars omitted. Also, it will be assumed that $x_r = d$ for some r. This may mean that the approximations in (7a) and (7b) are really to integrals over slightly smaller intervals, but the error can be made less than $\epsilon/2$ by taking m large enough.

From (5), $\theta < x$, $0 < x < \pi$, so $\pi/2 < c$. To show that $d < c$, suppose the contrary, and consider (7b) with c replaced by d and m by $m' \leqq m$. From (6a) and (6b), $\Delta_{k-1}x > \Delta_k x$ and $\Delta_k x > \Delta_k X$, if X is taken $\geqq d$. Then in corresponding terms of the sums in (7a) and (7b), $1 - p_k > P_k - 1$. Thus

$$\int_0^c (1-p)\cos^2\theta dx \geqq \int_d^\pi (p-1)\cos^2\theta dx.$$

When the integral over (c, d) is added, (5) is contradicted.

Now (6a) and (6b) show

$$\text{(8a)}\quad \Delta_{k-1}x > \Delta_k x, \quad x < d,$$

$$\text{(8b)}\quad \Delta_{k-1}X < \Delta_k X, \quad \text{and}$$

$$\text{(8c)}\quad \Delta_k x > \Delta_k X, \quad x < d.$$

The approximations (7a) and (7b) can be combined to give

$$\text{(9)}\quad -2\epsilon < \sum_{k=1}^m [(p_k - 1)\Delta_k x + (P_k - 1)\Delta_k X]\cos^2\theta_k < 2\epsilon.$$

Now if any term in (9) is negative, so are its predecessors. For this can only occur when $x_k \leqq d$. Suppose $D_k = (p_k - 1)\Delta_k x + (P_k - 1)\Delta_k X < 0$. From the monotony of $p(x)$,

$$(p_{k-1} - 1) = p_k - 1 - \alpha_1, \qquad (P_{k-1} - 1) = P_k - 1 + \alpha_2;$$

from (8a) and (8b),

$$\Delta_{k-1}x = \Delta_k x + \beta_1 \quad \text{and} \quad \Delta_{k-1}X = \Delta_k X - \beta_2,$$

where the α_i and β_i are positive. Then

$$\begin{aligned} D_{k-1} &= (p_{k-1} - 1)\Delta_{k-1}x + (P_{k-1} - 1)\Delta_{k-1}X \\ &= [(p_k - 1)(\Delta_k x + \beta_1) + (P_k - 1)(\Delta_k X - \beta_2)] \\ &\quad - \alpha_1(\Delta_k x + \beta_1) + \alpha_2(\Delta_k X - \beta_2). \end{aligned}$$

The square bracket is $<D_k<0$, because the negative term is multiplied by a larger positive number, and the positive term by a smaller, than before. Since $\Delta_k x > \Delta_k X$ by (8c), the concavity of $p(x)$ shows $p_k - p_{k-1} > P_{k-1} - P_k$, so $\alpha_1 > \alpha_2$. Hence $D_{k-1} < D_k$.

Now suppose (9) written in the form $-2\epsilon < \sum_{k=1}^m a_k b_k < 2\epsilon$, where $b_k = \cos^2\theta_k$. Let $\lambda = \sec^2\theta_r$, where the rth term is the last negative term in (9), and $S_m = \sum_{k=1}^m a_k$. Then

$$S_m - 2\lambda\epsilon < \sum_{k=1}^m a_k(1 - \lambda b_k) < S_m + 2\lambda\epsilon.$$

The choice of λ shows $\lambda b_k \geqq 1$ if $k \leqq r$ and <1 if $k > r$. Thus $(1 - \lambda b_k)$ is negative if a_k is negative and positive or zero otherwise. Hence the sum in the inequality is positive. Since S_m approximates $\int_0^\pi (p-1)dx$, the theorem is proved.

If the original independent variable is reintroduced, Theorem 1 becomes $\int_a^b [p - \pi^2/(b-a)^2]dx \geqq 0$. The proof of the main theorem uses this form.

Let the zeros of $y'(x)$ be $x_0 < x_1 < \cdots < x_n$.

Lemma 1. $x_k - x_{k-1} > x_{k+1} - x_k$, $k = 1, 2, \cdots, n$.

Proof. If (5) is formed for these intervals, the functions p that appear are the original functions multiplied by $(x_k - x_{k-1})^2/\pi^2$ and $(x_{k+1} - x_k)^2/\pi^2$ respectively. Then unless the lemma is true, the value of $p - 1$ is greater everywhere in the second integral than in the first, so the integrals cannot both be zero.

Proof of the main theorem. Application of Theorem 1 to the successive pairs of zeros of $y'(x)$ gives

$$\int_{x_0}^{x_n} p(x)dx \geqq \sum_{k=1}^m \pi^2/(x_k - x_{k-1}).$$

Lemma 3 of [3], due to E. Makai, Jr., then shows that

$$\int_{x_0}^{x_n} p(x)dx \geqq n^2\pi^2/(x_n - x_0).$$

Then if $x_0 = a$, $x_n \geqq b$. Equations (3) and (4), with θ replaced by $\theta - \alpha$, show that the zeros of the derivative of any other solution of (1) must alternate with the $\{x_k\}$. This completes the proof.

LEMMA 2. *Let $Y(x)$ be the solution of* (1) *that satisfies* $Y(0)=0$, $Y'(0)=1$. *Then the next zero of $Y(x)$ precedes the first zero of $y'(x)$.*

PROOF. Suppose the independent variable transformed as before, and suppose $Y(\pi)>0$. Let $z=\sin x$. Then

$$0 \leqq \int_0^\pi [(Yz' - Y'z)^2/Y^2]dx$$

$$= \int_0^\pi (Yz' - Y'z)[d(z/Y)/dx]dx$$

$$= (Yz' - Y'z)(z/Y)\int_0^\pi - \int_0^\pi (pYz - Yz)(z/Y)dx.$$

Since z/Y has a limit at $x=0$, the integrated term is zero, so

$$0 \leqq \int_0^\pi (1-p)z^2dx = \int_0^\pi (1-p)dx + \int_0^\pi (p-1)\cos 2x\,dx.$$

But the first integral is negative by Theorem 1, and the second, since $p-1$ is concave, is not positive, by a lemma of E. Makai [**4**, pp. 370–371]. This contradiction establishes the lemma.

COROLLARY 1. *If the original independent variables are used, the nth zero of $Y(x)$ after $x_0=a$ precedes the nth zero of $y'(x)$. This follows from the application of Lemma 2 to successive pairs of consecutive zeros.*

COROLLARY 2. *From Corollary* 1 *and the fact that the zeros of two solutions of* (1) *interlace, it follows that if $\{x_k\}$, $x_0<x_1<x_2< \cdots <x_n$, are $n+1$ consecutive zeros of $y'(x)$, there will be solutions of* (1) *with $n+1$ zeros in (x_0, x_n); since a zero of a solution separates two zeros of its derivative there can be no more than $n+1$ zeros of a solution in (x_0, x_n).*

The results are the best possible, in the sense that, given a positive integer n, we can find $p(x)$ such that some solution of (1) has $n+1$ zeros in (a, b), where b is determined by (2). The argument uses only known properties of Bessel functions, with $p(x)=cx^{m-2}$, where m is easily chosen in the interval (2, 3).

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