ON A SYSTEM OF INTEGRAL INEQUALITIES¹

S. G. DEO² AND M. G. MURDESHWAR

ABSTRACT. The present note obtains a vector extension and a further generalization of Bihari's Lemma on an integral inequality. The inequality proved can be used in the study of the component-wise behaviour of solutions of differential systems.

The theory of integral inequalities plays a vital role in the study of the stability properties of differential and integral equations. In particular, Bihari's integral inequality continues to be an effective tool to study sophisticated problems such as stability, boundedness and uniqueness of solutions [2], [3]. The statement of this lemma is as follows:

LEMMA (BIHARI). (i) *Let x and F be nonnegative continuous functions on $[0, \alpha)$, and*

(ii) *ω a positive nondecreasing continuous function on $(0, \infty)$ such that*

$$x(t) \leq k + M \int_0^t F(s)\omega(x(s))ds, \quad t \in [0, \alpha),$$

where k and M are positive constants. Then

$$x(t) \leq G^{-1}\left(G(k) + M \int_0^t F(s)ds\right), \quad t \in [0, \alpha_1),$$

where

$$G(u) = \int_{u_0}^u \frac{dt}{\omega(t)}, \quad u \geq u_0 > 0,$$

and G^{-1} is the inverse function of G and

$$\alpha_1 = \sup \left[t \in [0, \alpha) : G(\infty) \geq G(k) + M \int_0^t F(s)ds \right].$$

In this note we shall present (i) an extension to systems and (ii) a further generalization of the above lemma so that it will be effective in more delicate situations.

Received by the editors January 28, 1970.

AMS 1969 subject classifications. Primary 3490; Secondary 3442.

Key words and phrases. Integral inequality, Bihari's Lemma.

¹ This work was supported by NRC Canada Operating Grants A-3072 and A-3074.

² This author is on leave from Marathwada University, Aurangabad, India.

Let R^n denote the n -dimensional Euclidean space and $x_1, x_2, \dots, x_n$ are the components of an element $x \in R^n$. Assume that, whenever the subscript i occurs, it takes the values from 1 to n . Suppose that $\omega \in C[R^n, R]$ and the functions $\omega_i(u_1, u_2, \dots, u_n) > 0$ are nondecreasing in $u_1, u_2, \dots, u_n$. Assume that $\lim_{u_i \rightarrow \infty} \omega_i(u_1, u_2, \dots, u_i, \dots, u_n)$, as each component except u_i tends to ∞ , exists and is denoted by $\tilde{\omega}_i(u_i)$. Now we prove the following theorem.

THEOREM 1. *If*

- (i) x_i, F_i are nonnegative continuous functions on $[0, \alpha)$, and
- (ii) the vector functions ω and $\tilde{\omega}$, as described above, are such that

$$(1) \quad x_i(t) \leq k_i + M_i \int_0^t F_i(s) \omega_i(x(s)) ds, \quad t \in [0, \alpha),$$

where k_i and M_i are positive constants, then

$$(2) \quad x_i(t) \leq G_i^{-1} \left(G_i(k_i) + M_i \int_0^t F_i(s) ds \right), \quad t \in [0, \bar{\alpha}),$$

where

$$(3) \quad G_i(u_i) = \int_{u_{i0}}^{u_i} \frac{dy_i}{\tilde{\omega}_i(y_i)}, \quad u_i \geq u_{i0} > 0,$$

and G_i^{-1} is the inverse function of G_i and $[0, \bar{\alpha}) = \min_i [0, \alpha_i)$ where

$$\alpha_i = \sup \left[t \in [0, \alpha) : G_i(\infty) \geq G_i(k_i) + M_i \int_0^t F_i(s) ds \right].$$

PROOF. Let the right-hand side of (1) be denoted by $z_i(t)$. Then

$$z_i'(t) = M_i F_i(t) \omega_i(x(t)) \leq M_i F_i(t) \tilde{\omega}_i(z_i(t))$$

which in view of (3) reduces to

$$(4) \quad dG_i(z_i)/dt \leq M_i F_i(t).$$

Now the conclusion (2) follows on integrating (4) from 0 to t . The domain of the function G^{-1} is obvious (cf. [3]).

REMARK 1. It is easy to prove the following inequality. Let the assumptions (i) and (ii) of Theorem 1 hold. If the inequality (1) is replaced by

$$x_i(t) \geq k_i - M_i \int_0^t F_i(s) \omega_i(x(s)) ds, \quad t \in [0, \alpha),$$

then

$$x_i(t) \geq G_i^{-1} \left[G_i(k_i) - M_i \int_0^t F_i(s) ds \right]$$

on $[0, \bar{\alpha}) = \min_i [0, \alpha_i)$, where

$$\alpha_i = \sup \left[t \in [0, \alpha) : G_i(k_i) > M_i \int_0^t F_i(s) ds \right].$$

This extends the inequality proved in [4]. Now we obtain a generalization of the same lemma.

THEOREM 2. *Let the assumptions (i) and (ii) of Theorem 1 hold and f be a strictly increasing and continuous function on $[0, \infty)$ such that $f(0) = u_{i0}$, $f(\infty) = \infty$, and*

$$(5) \quad f(x_i(t)) \leq k_i + M_i \int_0^t F_i(s) \omega_i(x(s)) ds, \quad t \in [0, \alpha),$$

where k_i and M_i are positive constants. Then

$$(6) \quad f(x_i(t)) \leq G_i^{-1} \left(G_i(k_i) + M_i \int_0^t F_i(s) ds \right), \quad t \in [0, \bar{\alpha}),$$

where

$$(7) \quad G_i(u_i) = \int_{u_{i0}}^{u_i} \frac{dy_i}{\bar{\omega}_i(f^{-1}(y_i))}, \quad u_i \geq u_{i0} > 0,$$

and G_i^{-1} is the inverse function of G_i and $[0, \bar{\alpha}) = \min_i [0, \alpha_i)$ where

$$\alpha_i = \sup \left[t \in [0, \alpha) : G_i(\infty) \geq G_i(k_i) + M_i \int_0^t F_i(s) ds \right].$$

PROOF. Let the right-hand side of (5) be denoted by $z_i(t)$. Then, it is easy to observe that

$$M_i F_i(t) \omega_i(x(t)) \leq M_i F_i(t) \bar{\omega}_i(f^{-1}(z_i(t)))$$

which in view of (7) can be written as

$$dG_i(z_i)/dt \leq M_i F_i(t).$$

The conclusion (6) is now clear.

REMARK 2. The inequality (5) is much stronger than the inequality (1). In fact, if $f^{-1} = g$ and the function g is subadditive then the inequality (5) takes the form

$$x_i(t) \leq \eta_i + g \left(M_i \int_0^t F_i(s) \omega_i(x(s)) ds \right).$$

This inequality is much more general and it includes the inequalities proved in [1], [2] as special cases.

REMARK 3. It has been established in [3] that the comparison method provides a powerful tool for obtaining bounds for solutions and it is a unified approach to many similar problems. The comparison equation employed in [2], [3] is a scalar one. It is now possible to extend these results where a scalar equation can be replaced by a system. This approach will facilitate the study of the componentwise behaviour of a solution of a differential system. The details are omitted.

REFERENCES

1. R. Bellman, *The stability of solutions of linear differential equations*, Duke Math. J. **10** (1943), 643–647. MR **5**, 145.
2. I. Bihari, *A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations*, Acta. Math. Acad. Sci. Hungar. **7** (1956), 81–94. MR **18**, 38.
3. F. Brauer, *Bounds for solutions of ordinary differential equations*, Proc. Amer. Math. Soc. **14** (1963), 36–43. MR **26** #397.
4. C. E. LANGENHOP, *Bounds on the norm of a solution of a general differential equation*, Proc. Amer. Math. Soc. **11** (1960), 795–799. MR **22** #12260.

UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA, CANADA