TRACE-CLASS AND CENTRALIZERS OF AN H^*-ALGEBRA[1]

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ABSTRACT. Let A be a proper H^*-algebra. Let $\tau(A) = \{xy \mid x, y \in A\}$, let $R(A)$ be the set of all bounded linear operators S on A such that $S(xy) = (Sx)y$ for all $x, y \in A$ and let $C(A)$ be the closed subspace of $R(A)$ generated by the operators of the form $La: x \rightarrow ax$, $a \in A$. It is shown that $\tau(A)$ can be identified with the space of all bounded linear functionals on $C(A)$ and that $R(A)$ is the dual of $\tau(A)$. Also it is proved that $\tau(A)$ is a Banach algebra.

This paper is a continuation of [4]. It generalizes Theorems 2, 3 on pp. 47, 48 of [6]. These theorems state that the trace-class (τc) of operators on a Hilbert space H can be identified with the set of all bounded linear functionals on the ideal of completely continuous operators on H and that the dual of (τc) is essentially the set of all bounded operators on H.

A consequence of the generalization of the first fact is the completeness of the trace-class $\tau(A)$ for an arbitrary proper H^*-algebra A.

Let A be a proper H^*-algebra [1]. We define $R(A)$ to be the set of all right centralizers on A [2] ($R(A)$ is the set of all bounded operators S on A such that $S(xy) = (Sx)y$ for all $x, y \in A$). We define $C(A)$ to be the closed subspace of $R(A)$ generated by the operators $La: x \rightarrow ax$, $a \in A$, acting on A; $C(A)$ is the closure in the operator norm of the set $\{La \mid a \in A\}$.

For each $a \in \tau(A)$ consider the mapping f_a on $C(A)$ defined by $f_a(S) = \operatorname{tr}(Sa)$, $S \in C(A)$. Then $|f_a(S)| = |\operatorname{tr}(Sa)| \leqq \tau(Sa) \leqq \|S\| \tau(a)$ for each $S \in C(A)$ [4, Lemma 5] and so f_a is a bounded linear functional on $C(A)$ with $\|f_a\| \leqq \tau(a)$. The converse inequality also holds:

LEMMA. *If $a \in \tau(A)$ then the mapping $f_a: S \rightarrow \operatorname{tr}(Sa)$, defined on $C(A)$, is a bounded linear functional and $\|f_a\| = \tau(a)$.*

PROOF. We only need to prove that $\tau(a) \leqq \|f_a\|$. Let $\{e_n\}$ and $\{\mu_n\}$ be respectively a sequence of mutually orthogonal projections and a sequence of positive numbers such that $[a] = \sum_{n=1}^{\infty} \mu_n e_n$ and

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$[a]e_k = e_k[a] = \mu_k e_k$ for each k (see the definition of $[a]$ in [4] (before Lemma 2)).

For each n let $c_n = \sum_{k=1}^n e_k$; then $Lc_n \in C(A)$, $\|Lc_n\| = 1$ and

$$f[a](Lc_n) = \operatorname{tr}(Lc_n([a])) = \operatorname{tr}(c_n[a]) = \operatorname{tr}\left(\sum_{k=1}^n e_k[a]\right)$$
$$= \operatorname{tr}\left(\sum_{k=1}^n \mu_k e_k\right) = \sum_{k=1}^n \mu_k \operatorname{tr}(e_k) = \sum_{k=1}^n \mu_k(e_k, e_k).$$

Thus we have:

$$\sum_{k=1}^n \mu_k(e_k, e_k) = |f[a](Lc_n)| \leqq \|f[a]\| \cdot \|Lc_n\| = \|f[a]\|.$$

It follows that

$$\tau(a) = \operatorname{tr}[a] = \sum_{k=1}^\infty ([a]e_k, e_k) = \sum_{k=1}^\infty \mu_k(e_k, e_k) \leqq \|f[a]\|,$$

and so we only have to show now that $\|f[a]\| \leqq \|f_a\|$.

Let W be the partial isometry associated with a (see definition of W in [4] after Lemma 2). Then W, W^* are right centralizers of norm 1, $W[a] = a$, $W^*a = [a]$ and we have for each $S \in C(A)$:

$$|f[a](S)| = |\operatorname{tr}(S[a])| = |\operatorname{tr}(SW^*a)| = |f_a(SW^*)|$$
$$\leqq \|f_a\| \cdot \|SW^*\| \leqq \|f_a\| \cdot \|S\| \cdot \|W^*\| \leqq \|f_a\| \cdot \|S\|.$$

This simply means that $\|f[a]\| \leqq \|f_a\|$. . . .

THEOREM 1. *Each bounded linear functional on $C(A)$ is of the form f_a for some $a \in \tau(A)$. The correspondence $a \leftrightarrow f_a$ is an isometric isomorphism between $\tau(A)$ and $C(A)^*$.*

PROOF. The fact that the mapping $a \to f_a$ is one-to-one is easy to verify: if $f_a = f_b$ then the equality $(a, x^*) = \operatorname{tr}(xa) = \operatorname{tr}(Lx(a)) = f_a(Lx)$ $= f_b(Lx) = \operatorname{tr}(xb) = (b, x^*)$ holds for each $x \in A$, and this means that $a = b$.

To prove that the mapping $a \to f_a$ is onto, we use Riesz' Theorem about linear functionals on a Hilbert space. Let f be a bounded linear functional on $C(A)$. Then the mapping $x \to f(Lx)$ is a bounded linear functional on A since $|f(Lx)| \leqq \|f\| \cdot \|Lx\| \leqq \|f\| \cdot \|x\|$ (note that $\|Lx\| \leqq \|x\|$ for each $x \in A$). Hence there exists $a \in A$ such that $f(Lx)$ $= (x, a^*)$ for all $x \in A$. Then $f(Lx) = (x, a^*) = \operatorname{tr}(ax) = \operatorname{tr}(xa)$ $= \operatorname{tr}(Lx(a))$ for each $x \in A$. If we could show that $a \in \tau(A)$ this would imply that f is of the form $f = f_a$.

So let $\{e_n\}$, $\{\mu_n\}$ and W be as in the above lemma ($a^*a=\sum_n \mu_n^2 e_n$, $[a]=\sum_n \mu_n e_n$, $W[a]=a$, $W^*a=[a]$ and $Wx=\sum_n \mu_n^{-1}ae_nx$, $W^*x=\sum_n \mu_n^{-1}e_na^*x$ for each $x\in A$) (see discussion following Lemma 2 in [4]). For each n let $c_n=\sum_{k=1}^n e_k$, $d_n=\sum_{k=1}^n \mu_k^{-1}e_ka^*$. Then

$$Ld_n(x) = d_nx = \sum_{k=1}^{n} \mu_k^{-1}e_ka^*x = c_n\sum_{k=1}^{\infty} \mu_k^{-1}e_ka^*x = Lc_n(W^*x)$$

and so $\|Ld_n(x)\|=\|Lc_n(W^*x)\|\leqq\|Lc_n\|\cdot\|W^*\|\cdot\|x\|\leqq\|x\|$, which means that $\|Ld_n\|\leqq 1$. Then we have for each n:

$$\begin{aligned}\sum_{k=1}^{n}\mu_k(e_k, e_k) &= \sum_{k=1}^{n}\mu_k \operatorname{tr}(e_k) = \left|\operatorname{tr}\left(\sum_{k=1}^{n}\mu_k e_k\right)\right| = \left|\operatorname{tr}\left(\sum_{k=1}^{n}\mu_k^{-1}e_ka^*a\right)\right| \\ &= |\operatorname{tr}(d_na)| = |\operatorname{tr}(Ld_n(a))| = |f(Ld_n)| \\ &\leqq \|f\|\cdot\|Ld_n\| = \|f\|.\end{aligned}$$

This simply means that the series $\sum_{k=1}^{\infty}\mu_k(e_k, e_k)$ converges, $b=\sum_k(\mu_k)^{1/2}e_k$ belong to A and $[a]=b^2$ belongs to $\tau(\ )$.

COROLLARY. *$\tau(A)$ is a Banach algebra in the norm $\tau(A)$.*

PROOF. $\tau(A)$ is complete since it is isometric to the dual of $C(A)$ (see 8A in [3]).

THEOREM 2. *For each $S\in R(A)$ the mapping $f_S: x\rightarrow \operatorname{tr}(Sx)$, $f_S(x)=\operatorname{tr}(Sx)$, is a bounded linear functional on $\tau(A)$ such that $\|f_S\|=\|S\|$. Conversely, each bounded linear functional on $\tau(A)$ is of the form f_S for some $S\in R(A)$. Thus $R(A)$ is isomorphic and isometric to the space $\tau(A)^*$ of bounded linear functionals on $\tau(A)$.*

PROOF. Inequality $|\operatorname{tr}(Sx)|\leqq\|S\|\cdot\tau(x)$ (Lemma 5 in [4]) implies that f_S is a bounded linear functional on $\tau(A)$ and that $\|f_S\|\leqq\|S\|$. To establish the converse inequality we select for a given $\epsilon>0$ a member $a\in A$ with $\|a\|=1$ such that $\|S^*\|^2-\epsilon\leqq\|S^*a\|^2$ and then consider:

$$\begin{aligned}\|S^*\|^2-\epsilon &\leqq \|S^*a\|^2 = (S^*a, S^*a) = |(SS^*a, a)| = |\operatorname{tr}(SS^*(aa^*))| \\ &= |\operatorname{tr}(S(S^*aa^*))| = |f_S(S^*aa^*)| \\ &\leqq \|f_S\|\cdot\tau(S^*(aa^*)) \leqq \|f_S\|\cdot\|S^*\|\cdot\tau(aa^*) \\ &= \|f_S\|\cdot\|S^*\| \operatorname{tr}(aa^*) \\ &= \|f_S\|\cdot\|S^*\|\cdot\|a\|^2 = \|f_S\|\cdot\|S^*\|.\end{aligned}$$

Since this is valid for each $\epsilon>0$ we have: $\|S\|=\|S^*\|\leqq\|f_S\|$.

Now let $f\in\tau(A)^*$. For a fixed $a\in A$ the linear functional $g_a: x\rightarrow f(ax)$

is bounded on A: $|g_a(x)|=|f(ax)|\leqq\|f\|\tau(ax)\leqq\|f\|\cdot\|a\|\cdot\|x\|$ (Corollary 4 in [4]) and so $\|g_a\|\leqq\|f\|\cdot\|a\|$. Hence there is a member b_a of A such that $f(ax)=g_a(x)=(x, b_a^*)=\operatorname{tr}(b_a x)$ for each $x\in A$ and $\|b_a\|\leqq\|f\|\cdot\|a\|$. Define the mapping S on A by setting $Sa = b_a$ for each $a\in A$. Then S is a bounded linear operator on S such that $\|S\|\leqq\|f\|$ and $\operatorname{tr}((Sa)x)=f(ax)$ for all $x, a\in A$. Let us show that S is a right centralizer. If $u, v\in A$ then we have for each $x\in A$:

$$\begin{aligned}(S(uv), x^*) &= \operatorname{tr}(xS(uv)) = \operatorname{tr}((S(uv))x) = f((uv)x) = f(u(vx))\\ &= \operatorname{tr}(Su(vx)) = (Su, (vx)^*) = (Su, x^*v^*) = ((Su)v, x^*).\end{aligned}$$

Thus $S(uv)=(Su)v$. . . .

It follows also that $f=f_S$: if $a=uv$ for some $u, v\in A$ then $f(a) =f(uv)=\operatorname{tr}((Su)v)=\operatorname{tr}(S(uv))=\operatorname{tr}(Sa)=f_S(a)$.

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