SURFACES OF FINITE DOMAIN RANK

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ABSTRACT. In this paper it is shown that a surface has finite domain rank if and only if it is homeomorphic to a domain of a compact or closed surface.

1. Introduction. In this paper, the main result of [1] is extended to surfaces (connected 2-manifolds) with boundary. The principal problem is to find necessary and sufficient conditions which assure that a surface of finite genus with boundary has finite domain rank. As in [1], a surface M without boundary is called closed if M is compact; otherwise, M is called open. A compact surface will always mean a surface with boundary which is compact.

2. Proof of the main results. As in [1], if M is a compact surface, let $g(M)$ denote the *genus* of M . For any surface M , the *orientation number* of M , denoted by $\rho(M)$, is defined by $\rho(M) = 1$ if M is orientable and $\rho(M) = 0$ otherwise. For an arbitrary surface M , define the *genus* of M , $g(M)$, by

$$g(M) = \sup \{ g(K) : K \text{ is a compact surface, } K \subset M, \text{ and } \rho(K) = \rho(M) \}.$$

If $\rho(M) = 0$, define the *orientable genus* of M , $og(M)$, by

$$og(M) = \sup \{ g(K) : K \text{ is a compact surface, } K \subset M, \rho(K) = 1 \}.$$

Clearly, $g(M) = g(\text{int } M)$. Therefore it follows from Theorem 2.8 of [1] that $\text{int } M$ has finite domain rank if and only if $g(M)$ is finite. However, it is not necessarily true that $\text{DR}(M)$ (domain rank of M) is finite when $g(M)$ is finite. Let

$$M = R^2 - \left(\bigcup_{i=1}^{\infty} G_i \right)$$

where

$$G_i = \{ (x, y) \in R^2 : (x - i)^2 + y^2 < (1/4)^2 \}.$$

Then M is a surface with boundary, $g(M) = 0$, but $\text{DR}(M)$ is clearly infinite.

It follows from Theorem 2.8 of [1] that any surface M with finite genus is homeomorphic to a subset of some closed surface. However,

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it may not be homeomorphic to a domain (open, connected subset) of any closed or compact surface.

DEFINITION 2.1. Let M, N be surfaces, A a subset of M . An embedding f of A into N is called proper if $f(A \cap \text{bd } M) \subset \text{bd } N$ and $f(A \cap \text{bd } M) = f(A) \cap \text{bd } N$.

It is easily established that if M, N are surfaces, D a domain of M , and f a proper embedding of D into N , then $f(D)$ is a domain of N .

Let M be a closed surface; $\{B_j\}_{j=1}^t$ a disjoint collection of 2-cells in M , $t > 0$; p, q , and r nonnegative integers such that $p+q+r=t$; and if $q \neq 0$, let F be a finite set of points in M such that $F \subset (\bigcup_{j=p+1}^{p+q} \text{bd } B_j)$ and $F \cap \text{bd } B_j \neq \emptyset$, $p+1 \leq j \leq p+q$. Let

$A = \emptyset$ if $p=0$ and $A = \bigcup_{j=1}^p \text{int } B_j$ if $p > 0$;

$B = \emptyset$ if $q=0$ and $B = (\bigcup_{j=p+1}^{p+q} \text{int } B_j) \cup F$ if $q > 0$; and

$C = \emptyset$ if $r=0$ and $C = \bigcup_{j=p+q+1}^t B_j$ if $r > 0$.

Let the surface $M(p, q, r, F)$ be defined by

$$M(p, q, r, F) = M - (A \cup B \cup C).$$

REMARK. It is easily established that if $p' > p$ or $p' + q' > p + q$, then there is no proper embedding of $M(p', q', r', F')$ into $M(p, q, r, F)$.

THEOREM 2.2. If M is a surface, $\text{DR}(M) < \infty$, then M is homeomorphic to a domain of a compact or closed surface. Conversely, if M is a compact or closed surface of genus n with p boundary components, then

$$\text{DR}(M) = ((p+2)(p+1)/2)(n + [(n-1)/2](1 - \rho(M)) + 1).$$

PROOF. (a) Let M be a surface, $\text{DR}(M) < \infty$. If M is compact or closed, then the result follows trivially. If M is open, the result follows from Corollary 2.9 of [1]. Therefore, it may be assumed that M is a noncompact surface with boundary. Since $\text{DR}(M) < \infty$, clearly $\text{bd } M$ can have at most a finite number p , $p \geq 0$, of components which are 1-spheres. Since $\text{DR}(M) < \infty$, it follows from [1] that $\text{DR}(\text{int } M) < \infty$ and therefore $g(M) = g(\text{int } M) < \infty$. Let N be a closed surface such that $g(N) = g(M)$ and $\rho(N) = \rho(M)$. There exists a sequence $\{L_j\}_{j=1}^{\infty}$ of compact surfaces contained in M such that

(i) $M = \bigcup_{j=1}^{\infty} L_j$;

(ii) L_j contains all components of $\text{bd } M$ which are 1-spheres, $j \geq 1$;

(iii) $L_j \subset iL_{j+1}$, $j \geq 1$, where iL_{j+1} denotes the point set interior of L_{j+1} in M ;

(iv) each component of $L_j \cap \text{bd } M$ is a 1-cell or a 1-sphere, $j \geq 1$; and

(v) $L_j \equiv N(s(j))$, where $N(s(j)) = N(s(j), 0, 0, \emptyset)$ and $s(j) > p$, $j \geq 1$ ($\equiv$ denotes topological equivalence).

For each $j \geq 1$, let $q(j)$ be the number of components of $\text{bd } L_j$ which have nonempty intersection with $\text{bd } M$, but which do not meet $\text{bd } M$ in a 1-sphere. Without loss of generality, we may assume that $q(j) = s$, s a fixed integer for all $j \geq 1$ or that $q(j+1) > q(j)$ for all $j \geq 1$. If the latter were true, there would exist a sequence of domains $\{D_j\}_{j=1}^\infty$, $iL_j = D_j \equiv N(p, q(j), r(j), F_j)$. Since $\text{DR}(M) < \infty$, it follows that there exist integers u, v , $1 \leq u < v$, such that D_u and D_v have a common generator. Since there is a proper embedding f_v of L_v (considered as a subset of M) into M such that $f_v(L_v) \subset iL_v$, it follows that there exists a proper embedding g of D_v into D_u . However, this contradicts the remark above. Therefore, the former must always hold. Since $L_j = N(s(j))$ and $q(j) = s$ for all $j \geq 1$, it follows that for all $j \geq 1$, $L_{j+1} - iL_j$ is a disjoint collection $\{M'_i\}_{i=1}^{\alpha(j)}$ of compact surfaces of genus zero such that $M'_i \cap L_j$ is a 1-sphere or 1-cell for $1 \leq i \leq \alpha(j)$. Therefore, it follows that any proper embedding f_j of L_j into $N(p+s)$ can be extended to a proper embedding f_{j+1} of L_{j+1} into $N(p+s)$. Since there does exist a proper embedding f_1 of L_1 (considered as a subset of M) into $N(p+s)$, a proper embedding f of M into $N(p+s)$ is easily constructed. Since $f(M)$ is a domain of $N(p+s)$, the result is established.

(b) Let $Q(n, \rho, p)$ denote a closed or compact surface which has genus n , orientation number ρ and p boundary components. For $M = Q(n, \rho, p)$, set

$$\alpha(M) = ((p+2)(p+1)/2)(n + [(n-1)/2](1-\rho) + 1).$$

If $p=0$, $N=Q(n, \rho, 0)$, it follows from Theorem 2.8 of [1] that $\text{DR}(N) = \alpha(N)$. Thus we may assume that $M = Q(n, \rho, p)$, $p \geq 1$. Set

$$W = \{(k, \rho) : Q(k, \rho, 1) \text{ is embeddable in } M = Q(n, \rho, p)\}.$$

It follows from Lemmas 2.3 and 2.4 of [1] that W has

$$n + [(n-1)/2](1-\rho) + 1$$

elements. Let

$$V = \{(s, t) : s, t \text{ are nonnegative integers, } 0 \leq s+t \leq p\}.$$

For each pair $(k, \rho) \in W$ and $(s, t) \in V$ let

$$D[k, \rho; s, t] = Q_p^k(s, t, 1, F)$$

where $Q_p^k = Q(k, \rho, 0)$ and F is a finite set of points with exactly t elements. Then $D[k, \rho; s, t]$ is homeomorphic to a proper domain of M . If a domain G generates both $D[k, \rho; s, t]$ and $D[k', \rho'; s', t']$, then

it follows from Lemma 2.5 of [1] and the above remark that $k=k'$, $\rho=\rho'$, $s=s'$, and $t=t'$. Therefore $\text{DR}(M) \geq \alpha(M)$.

Now suppose that D is a proper domain of $M=Q(n, \rho, p)$. Then $D=\bigcup_{j=1}^{\infty} L_j$ where for $j \geq 1$, L_j is a compact surface contained in M , $L_j \subset iL_{j+1}$, $L_j \cap \text{bd } M$ is empty or each component of $L_j \cap \text{bd } M$ is a 1-sphere or 1-cell, and $L_j=Q(k, \rho, r_j)$, $(k, \rho) \in W$. For each pair $(k, \rho) \in W$ and $(s, t) \in V$, let $Q_p^k=Q(k, \rho, 0)$ with $\{B_j\}_{j=1}^{s+t}$ a disjoint collection of 2-cells in Q_p^k (this is an empty collection if $s+t=0$). Let $A=\{0\} \cup \{1/n: n \text{ is a positive integer}\}$ with the induced topology of the real line. Let $A_0 \equiv A$, $A_0 \subset Q_p^k - (\bigcup_{j=1}^{s+t} B_j)$ and if $t \neq 0$, let $A_j \equiv A$, $A_j \subset \text{bd } B_j$, for $s+1 \leq j \leq s+t$. Set

$$G(k, \rho; s, t) = Q_p^k - \left[\left(\bigcup_{j=1}^{s+t} \text{int } B_j \right) \cup \left(\bigcup_{j=s+1}^{s+t} A_j \right) \cup A_0 \right].$$

For $j \geq 1$, let $\text{ed } L_j = L_j - iL_j$ (iL_j denotes the point set interior of L_j relative to M). For any $j \geq 1$, let L_j^* denote the space obtained by attaching $\text{ed } L_j \times [0, 1]$ to L_j via the map $c: \text{ed } L_j \times \{0\} \rightarrow L_j$ where $c(x, 0)=x$, $x \in \text{ed } L_j$. Consider $\text{ed } L_j \times [0, 1]$ and L_j as embedded in L_j^* in the usual way. Since $L_j \subset iL_{j+1}$, there is an embedding $f_j: L_j^* \rightarrow iL_{j+1}$ such that $f_j(x)=x$, $x \in L_j$, and

$$f_j(\text{ed } L_j \times [0, 1]) \cap \text{bd } M = f_j((\text{ed } L_j \cap \text{bd } M) \times [0, 1]).$$

Let $L'_j = f_j(L_j^*)$. For fixed j , let r be the number of components of $\text{bd } M$ which are contained in L_j and let $\{S_q\}_{q=1}^r$ be the components of $\text{bd } L_j$ which have nonempty intersection with $\text{bd } M$ but which do not intersect in a 1-sphere. Then it follows from the remark preceding Theorem 2.2 that $0 \leq r+s \leq p$. If $s \geq 1$, then for $1 \leq q \leq s$, let E_q be a component of $L_j \cap \text{bd } M$ which is a 1-cell, $E_q \subset S_q$, and let E'_q be the 1-cell component of $L'_j \cap \text{bd } M$ such that $E_q \subset E'_q$. Then there exist subsets $A'_0, A'_1, \dots, A'_s$ such that $A'_0 \subset (iL'_j - L_j) \cap \text{int } M$ and for $1 \leq q \leq s$, $A'_q \subset (iL'_j \cap E'_q) - E_q$. Since $L_j=Q(k, \rho, r_j)$, $(k, \rho) \in W$, for properly chosen $\{A'_i\}_{i=0}^s$, the domain G_j of M defined by $G_j = L'_j - (\text{ed } L'_j \cup (\bigcup_{q=0}^s A'_q))$ is homeomorphic to $G(k, \rho; r, s)$. Therefore $D = \bigcup_{j=1}^{\infty} G_j$ where for all $j \geq 1$, $G_j \subset G_{j+1}$. Since there are precisely $\alpha(M)$ choices for the domains $\{G_j\}_{j=1}^{\infty}$ (up to homeomorphism), it follows that $\text{DR}(M) \leq \alpha(M)$ and therefore $\text{DR}(M) = \alpha(M)$ and the result is established.

REFERENCE

1. R. J. Tondra, *Characterization of connected 2-manifolds without boundary which have finite domain rank*, Proc. Amer. Math. Soc. 22 (1969), 479-482.