ON THE COHOMOLOGY CHERN CLASSES OF THE K-THEORY CHERN CLASSES

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ABSTRACT. Let ξ be a vector bundle over a finite complex and $\gamma^i \xi$ its i th- K theory Chern class. We first show that

$$c_n \gamma^i \xi = (i-1)! S(n, i) c_n \xi + \text{decomposables},$$

where $S(n, i)$ is a Stirling number of the second kind. We apply this result to show that certain multiples of the e -invariant of a map $S^{2m-1} \rightarrow S^{2n}$ must always be integral.

1. On Chern classes of Chern classes. Let X be a finite complex and $\xi \in \tilde{K}(X)$. Then ξ has K -theory Chern classes [6] $\gamma^i \xi \in K^{2i}(X)$ which in turn have cohomology Chern classes $c_n \gamma^i \xi \in H^{2n}(X; \mathbf{Z})$.

Our objective is to study the problem of computing $c_n \gamma^i \xi$ as a function of $c_j \xi$. Contemplation of the universal example shows that there must be a universal formula expressing $c_n \gamma^i \xi$ as a polynomial in $c_0 \xi, \dots, c_n \xi$. As a step towards computing this polynomial we shall establish:

THEOREM 1.1. *Let X be a finite complex and ξ a vector bundle over X . Then we have*

$$c_n \gamma^i \xi = (i-1)! S(n, i) c_n(\xi) + \text{decomposables}.$$

(Here $S(n, i)$ is a Stirling number of the second kind. See the appendix for information about Stirling numbers.)

PROOF. It will clearly suffice to check the assertion in the universal example, i.e., with X a giant skeleton of BU and ξ the restriction of the canonical bundle. Let

$$\phi_n: S^{2n} \rightarrow BU$$

be a representative of a generator of $\pi_{2n}(BU) \cong \mathbf{Z}$. Recall [2] that

$$c_n(\xi_n) = (n-1)! \sigma_n$$

where ξ_n is the bundle induced by ϕ_n over S^{2n} and $\sigma_n \in H^{2n}(S^{2n}; \mathbf{Z})$ is the canonical generator.

Now recall that we have shown [5, 2.4, 2.5] that

$$\text{ch}^{2n} \gamma^i \xi_n = (i-1)! S(n, i) \sigma_n.$$

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However we also have the universal formula

$$\mathrm{ch}^{2n} \eta = (1/n!)(nc_n(\eta) + \text{decomposables})$$

for any bundle η over a finite complex. Since decomposables vanish in a sphere this specializes to

$$\mathrm{ch}^{2n} \gamma^i \xi_n = (1/(n-1)!)c_n \gamma^i \xi_n.$$

Equating these two expressions for $\mathrm{ch}^{2n} \gamma^i \xi_n$ leads to the formula

$$c_n \gamma^i \xi_n = (i-1)!S(n, i)(n-1)!\sigma_n$$

and recalling that $(n-1)!\sigma_n = c_n \xi_n$ we obtain

$$c_n \gamma^i \xi_n = (i-1)!S(n, i)c_n \xi_n.$$

If we let θ denote the universal bundle over BU then we have the universal formula $c_n \gamma^i \theta = A_{n,i} c_n \theta + \text{decomposables}$. Applying ϕ_n^* gives $c_n \gamma^i \xi_n = A_{n,i} c_n \xi_n$ and the result follows. $\square$

COROLLARY 1.2. *Let X be a finite complex and ξ a vector bundle over X . Then we have*

$$\mathrm{ch}^{2n} \gamma^i \xi = (i-1)!S(n, i)\mathrm{ch}^{2n} \xi + \text{decomposables}.$$

PROOF. This is implicit in the proof of (1.1) above and in addition follows easily from it. $\square$

COROLLARY 1.3. *Let X be a finite complex and ξ a vector bundle over X . Suppose that all products are zero in $\tilde{H}^*(X; \mathbb{Q})$, for example X is a suspension. Then*

$$\mathrm{ch}^{2n} \gamma^i \xi = (i-1)!S(n, i)\mathrm{ch}^{2n} \xi.$$

If moreover all products are zero in $\tilde{H}^*(X; \mathbb{Z})$ then

$$c_n \gamma^i \xi = (i-1)!S(n, i)c_n \xi. \quad \square$$

2. On the unstable e -invariant. Suppose given $f: S^{2m-1} \rightarrow S^{2n}$. Form the cofibration

$$S^{2n} \xrightarrow{i} S^{2n} \cup_f e^{2m} \xrightarrow{j} S^{2m}$$

yielding the exact sequence

$$0 \leftarrow \tilde{K}(S^{2n}) \xleftarrow{i^*} \tilde{K}(S^{2n} \cup_f e^{2m}) \xleftarrow{j^*} \tilde{K}(S^{2m}) \leftarrow 0.$$

Choose generators $\lambda, \mu \in K(S^{2n} \cup_f e^{2m})$ such that $j^* \xi_m = \lambda$, $i^* \mu = \xi_n$, where for any integer k , $\xi_k \in \tilde{K}(S^{2k}) \cong \mathbb{Z}$ is the canonical generator. We then have

$$\text{ch } \lambda = \sigma_m, \quad \text{ch } \mu = e(f)\sigma_m + \sigma_n: \quad e(f) \in \mathcal{Q},$$

where as previously $\sigma_k \in H^{2k}(S^{2k}; \mathbf{Z}) \cong \mathbf{Z}$ is the canonical generator. The rational number $e(f)$ is of course the e -invariant of f and its residue class in $\mathcal{Q}/\mathbf{Z}$ is independent of the choice of λ, μ [1], [7]. Our objective is to show that certain multiples of $e(f)$ must always be integral. To this end we introduce:

DEFINITION. Let $m, n, k \in \mathbf{Z}^+$, and set

$$g(m, n) = \text{g.c.d.} \left\{ k^m - k^n \right\}_{k > 1}.$$

In the appendix we will establish that

$$g(m, n) = \text{g.c.d.} \left\{ (k-1)! [S(m+1, k) - S(n+1, k)] \right\}_{1 < k < 1 + \max(m+1, n+1)}$$

where $m, n, k \in \mathbf{Z}^+$.

The main result of this section is the following.

THEOREM 2.1. *Let $f: S^{2m-1} \rightarrow S^{2n}$ and assume that $m \neq 2n$. Then $g(m-1, n-1)e(f) \in \mathbf{Z}$.*

This result is among results originally obtained by P. Hoffman in [3] extending a stable result of Adams [1] and Toda [7].

PROOF OF 2.1. Let us continue to employ the notations of the previous discussion. Then on general grounds we have

$$(A) \quad \gamma^i \mu = r_i \mu + s_i \lambda: \quad r_i, s_i \in \mathbf{Z}.$$

Moreover as we are assuming $m \neq 2n$ we obtain in view of (1.3)

$$(B) \quad \text{ch}^{2m} \gamma^i \mu = (i-1)! S(m, i) \text{ch}^{2m} \mu = (i-1)! S(m, i) e(f) \sigma_m,$$

$$(C) \quad \text{ch}^{2n} \gamma^i \mu = (i-1)! S(n, i) \text{ch}^{2n} \mu = (i-1)! S(n, i) \sigma_n,$$

while from (A) we obtain

$$(D) \quad \text{ch}^{2m} \gamma^i \mu = r_i \text{ch}^{2m} \mu + s_i \text{ch}^{2m} \lambda = r_i e(f) \sigma_m + s_i \sigma_m,$$

$$(E) \quad \text{ch}^{2n} \gamma^i \mu = r_i \text{ch}^{2n} \mu + s_i \text{ch}^{2n} \lambda = r_i \sigma_n.$$

Equating (C) and (E) leads to

$$(F) \quad r_i = (i-1)! S(n, i),$$

while equating (B) and (D) and employing (F) gives

$$(G) \quad (i-1)! [S(m, i) - S(n, i)] e(f) = s_i \in \mathbf{Z}.$$

Since i is arbitrary we conclude that

$$\text{g.c.d.} \left\{ (i-1)! [S(m, i) - S(n, i)] \right\} e(f) \in \mathbf{Z}.$$

But $g(m-1, n-1) = \text{g.c.d.} \{ (i-1)! [S(m, i) - S(n, i)] \}$ by the theorem from the appendix previously quoted. $\square$

Appendix : On Stirling numbers. Let us first recall one definition of the Stirling numbers of the second kind [4, II.7].

DEFINITION. Let n, k be positive integers. Then

$$S(n, k) = \frac{(-1)^k}{k!} \sum_{j=0}^k (-1)^j \binom{k}{j} j^n.$$

Note that

$$S(n, 1) = 1 = S(n, n), \quad S(n, k) = 0: \quad k > n,$$

and

$$S(n+1, k) = S(n, k-1) + kS(n, k),$$

and so from the two conditions above we obtain $S(n, k) \in \mathbb{Z}^+$; $k = 1, \dots, n$.

These numbers have a long history and many useful properties, see for example [4], [5].

The formula

$$(-1)^k k! S(n, k) = \sum_{j=0}^k (-1)^j \binom{k}{j} j^n$$

may be inverted to yield a formula expressing k^n as a function of $S(n, j)$, $j=1, \dots, n$. (This formula is often used to define the Stirling numbers.) The procedure runs as follows.

LEMMA 1. Let $m, n \in \mathbb{Z}^+$, then

$$\sum_{j=0}^m (-1)^j \binom{m}{j} (-1)^n \binom{j}{n} = \delta_{mn}$$

whenever $n \leq m$.

PROOF. Start from the identity $t^m = (1 - (1-t))^m$ and apply the binomial theorem twice to yield

$$t^m = \sum_{j=0}^m (-1)^j \binom{m}{j} (1-t)^j = \sum_{j, n=0}^m (-1)^j \binom{m}{j} (-1)^n \binom{j}{n} t^n$$

and the result follows from equating coefficients. $\square$

Return now to the formula

$$(-1)^k k! S(n, k) = \sum_{j=0}^k (-1)^j \binom{k}{j} j^n.$$

Multiplying by $(-1)^k \binom{i}{k}$ and summing from $k=0$ to $k=i$ we get

$$\sum_{k=0}^i k! \binom{i}{k} S(n, k) = \sum_{k=0}^i \sum_{j=0}^k (-1)^k \binom{i}{k} (-1)^j \binom{k}{j} j^n.$$

Interchanging the order of summation and applying Lemma 1 we obtain

$$\sum_{k=0}^i k! \binom{i}{k} S(n, k) = \sum_{j=0}^i \delta_{ij} j^n = i^n.$$

Hence we have established:

PROPOSITION 2. We have the formula $i^n = \sum_{k=0}^i k! \binom{i}{k} S(n, k)$, for all $n, i \in \mathbf{Z}^+$. $\square$

COROLLARY 3. Let n, m, k be positive integers. Then we have the formulas

$$k^m - k^n = \sum_{i=0}^k i! \binom{k}{i} [S(m, i) - S(n, i)]$$

and

$$(-1)^k k! [S(m, k) - S(n, k)] = \sum_{i=0}^k (-1)^i \binom{k}{i} (i^m - i^n).$$

PROOF. The first assertion follows from Proposition 2 and the second from the definitions. $\square$

COROLLARY 4. Let $n, m \in \mathbf{Z}^+$, then

$$g(m, n) = \text{g.c.d.}_{1 \leq k \leq 1 + \max(n, m)} \{k! [S(m, k) - S(n, k)]\}.$$

PROOF. Immediate from Corollary 3 and the fact that $S(r, s) = 0$ for $s > r$. $\square$

More generally than Corollary 4 we have:

PROPOSITION 5. Let $m, n \in \mathbf{Z}^+$. Then

$$(-1)^k (k-1)! [S(m, k) - S(n, k)] = \sum_{i=1}^k (-1)^i \binom{k-1}{i-1} (i^{m-1} - i^{n-1}),$$

$$k \in \mathbf{Z}^+.$$

PROOF. The result takes almost as long to state as to prove. Start with the formula

$$(-1)^k k! [S(m, k) - S(n, k)] = \sum (-1)^i \binom{k}{i} (i^m - i^n).$$

Using the binomial identity

$$\binom{k}{i} \frac{1}{k} = \binom{k-1}{i-1} \frac{1}{i}$$

rewrite it as

$$\begin{aligned} (-1)^k (k-1)! [S(m, k) - S(n, k)] &= \sum (-1)^i \binom{k-1}{i-1} \frac{1}{i} (i^m - i^n) \\ &= \sum (-1)^i \binom{k-1}{i-1} (i^{m-1} - i^{n-1}) \end{aligned}$$

which is the desired result. $\square$

Applying inversion and what not we obtain:

$$k^{m-1} - k^{n-1} = \sum_{i=1}^k (k-1)! \binom{k-1}{i-1} [S(m, i) - S(n, i)]$$

and hence we get:

COROLLARY 6. *Let $m, n \in \mathbf{Z}^+$, then*

$$g(m-1, n-1) = \text{g.c.d.}_{1 \leq k \leq 1+\max(n,m)} \{ (k-1)! (S(m, k) - S(n, k)) \},$$

$k \in \mathbf{Z}^+.$ $\square$

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